Chapter II

Complex Numbers, A Geometric View

1. Polar Form

The complex number system may be regarded as a numerical representation of the points in a plane (called the ***Argand Plane*** in this context). In the Argand Plane, one selects two points and calls them *O* (***origin***) and *U* (***unity***). The distance between *O* and *U* is chosen as the unit length. Then the location of any other point *P* in the plane is specified by polar coordinates $[r,\ \theta]$, where *r* is the distance from *O* to *P* and $\theta = \triangle UOP$. The angle θ is positive when measured counterclockwise and negative when measured clockwise. When θ is measured in radians, we will generally indicate that *P* has $[r,\ \theta]$ as polar coordinates by writing $P = re^{\theta i}$; this borrows a notation from Complex Variables courses. (See Supplementary Problem 6.)

For example, if the distance between *O* and *M* is two units and the angle (measured counterclockwise) from ray *OU* to ray *OM* is $\pi/4$, we write $M = 2e^{(\pi/4)i}$. [See Figure 1a.] Similarly, if the distance between *O* and *N* is ½ and the (clockwise) angle from ray *OU* to ray *ON* is $-2\pi/3$, we write $N = \frac{1}{2}e^{-2\pi i/3}$. [See Figure 1b.] The origin *O* has zero as its *r*-coordinate and any angle may be chosen as its angle; thus $O = 0\cdot e^{\theta i}$ for all real numbers θ.

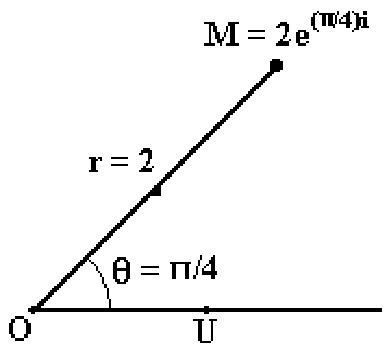

Figure 1a

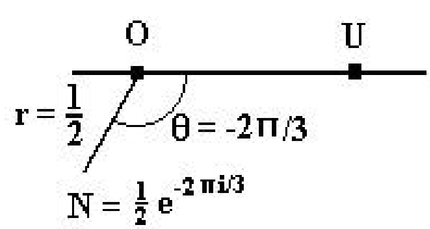

Figure 1b

2. Terminology

The expression $re^{\theta i}$ for a complex number is called its ***polar form***. The nonnegative real number *r* is called the ***absolute value*** or ***modulus*** or ***magnitude*** of *P* and we write $r = |P|$. The angle θ is called an ***argument*** of *P* and is denoted as arg *P*.

A fixed point *P* always has a unique nonnegative real number *r* as its absolute value (i.e., distance to *O*). On the other hand, *P* always has an infinite number of arguments since

$$re^{\theta i} = re^{(\theta \pm 2n\pi)i} \quad \text{for } n = 0,1,2,3,\ldots.$$

For example, some of the other representations for the point $M = 2e^{(\pi/4)i}$ of Figure 1a are $2e^{-7\pi i/4}$, $2e^{9\pi i/4}$, and $2e^{17\pi i/4}$. Also, e^{0i}, $e^{2\pi i}$, $e^{-2\pi i}$, and $e^{-4\pi i}$ are several of the representations of *U*.

The set of points in the Argand Plane is made into the ***Complex Number System*** by defining addition and multiplication as follows:

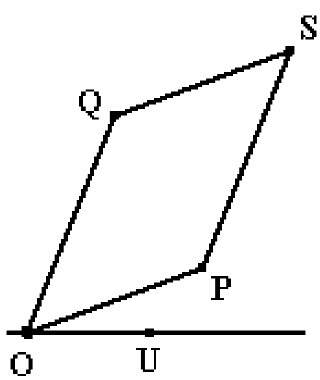

Figure 2

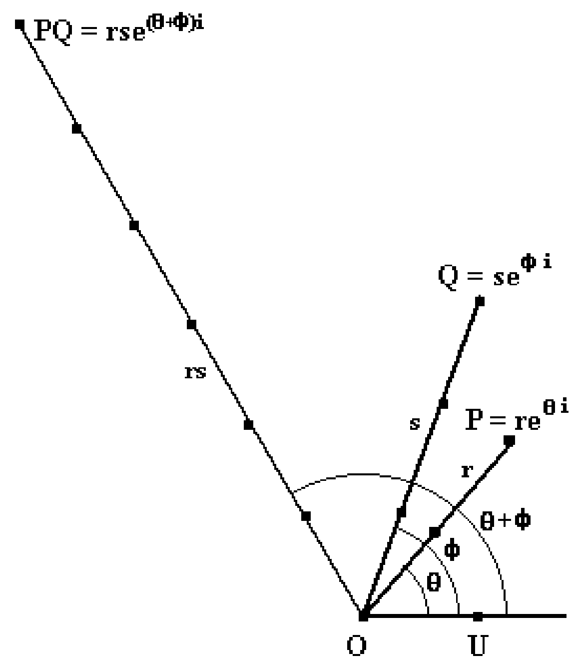

Figure 3

Sum. (Addition of complex numbers) *P* + *Q* is the point *S* such that the directed segment $\overrightarrow{PS}$ has the same magnitude and direction as $\overrightarrow{OQ}$. This means that the equation *S* = *P* + *Q* implies that the quadrilateral *OPSQ* is a parallelogram (See Figure 2) unless it collapses into a line segment.

Product. (Multiplication of complex numbers) If $P = re^{\theta i}$ and $Q = se^{\phi i}$, then $PQ = rse^{(\theta+\phi)i}$. Thus *PQ* is the point whose absolute value is the product of the absolute values of *P* and *Q* and whose arguments are the sums of an argument of *P* and an argument of *Q*. This is consistent with the fact that the angles θ and ϕ appear in the exponents. See Figure 3.

Exercises for Chapter II Sections 1 and 2

For all problems calling for a graph, make all diagrams neat and accurate with the unit of length at least half an inch. It may be helpful to use graph paper and a protractor. Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. Let $C = 2e^{-3\pi i/4}$, $D = 4e^{\pi i/6}$, $E = e^{2\pi i}$, and $F = e^{-2\pi i/3}$. Give the polar form (i.e., $re^{\theta i}$ form) for each of the following products.

   (a) CD; (b) CE; (c) CF; (d) C^2.

2. Give an alternate representation for $5e^{(\pi/3)i}$ using a negative argument and one using a positive argument different from $\pi/3$.

3. Let $A = 2e^{(5\pi/4)i}$ and $B = 3e^{(-\pi/3)i}$.

   (a) Give the polar form for AB.

   (b) Plot A, B, and AB.

   (c) Construct $S = A + B$ given that $OASB$ is a parallelogram.

4. Do as in problem 3 with $A = \sqrt{2}e^{(\pi/4)i}$ and $B = 2e^{(2\pi/3)i}$.

5. Let $G = 3e^{0i}$ and $H = 4e^{(\pi/2)i}$. Plot G, H, and $G + H$ and find $|G + H|$, i.e., the distance form O to $G + H$.

6. Let $K = 12e^{(\pi/2)i}$ and $L = 5e^{\pi i}$. Plot K, L, and $K + L$ and find $|K + L|$.

7. Let $M = 4e^{(7\pi/6)i}$ and $N = 2e^{(-\pi/6)i}$. Plot M, N, and $M + N$.

8. Let $M = 2\sqrt{3}e^{(\pi/4)i}$ and $N = 2e^{(3\pi/4)i}$. Plot M, N, and $M + N$.

9. The points $re^{\theta i}$ for which r is any nonnegative real number and θ is in $\{0, 2\pi, -2\pi, 4\pi, -4\pi, \ldots\}$ form the ray OU. Give similar geometric characterizations for the following sets of points:

   (a) The set R of all the points $re^{\theta i}$ with r any nonnegative real number and θ in $\{0, \pi, -\pi, 2\pi, -2\pi, 3\pi, \ldots\}$.

   (b) The set I of all the points $se^{\phi i}$ with s any nonnegative real number and ϕ in $\{\pi/2, -\pi/2, 3\pi/2, -3\pi/2, 5\pi/2, \ldots\}$.

10. Can every point P of the Argand Plane be expressed as $P = A + B$ with A in the set R of Problem 9 (a) and B in the set I of Problem 9 (b)? Explain.

11. Let $P = 6e^{(4\pi/9)i}$. Find the polar form of a point N such that $P + N = O$. Show P, O, and N in a diagram.

12. Let P be as in Problem 11. Find the polar form of a point M such that $PM = U$. Show P, U, and M in a diagram.

13. Let O be the origin and U be the unity in the Argand Plane and let P, Q, and R be any complex numbers. Verify that the complex number system has each of the following properties:

(a) Commutativity of addition: $P + Q = Q + P$.

(b) Associativity of addition: $(P + Q) + R = P + (Q + R)$. HINT: See Problem 13 of Chapter 1.

(c) Additive identity: $O + P = P + O = P$.

(d) Additive inverse: For each P there exists a complex number N such that $N + P = P + N = O$.

(e) Commutativity of multiplication: $PQ = QP$.

(f) Associativity of multiplication: $(PQ)R = P(QR)$.

(g) Zero multiplication: $OP = PO = O$.

(h) Multiplicative identity: $UP = PU = P$.

(i) Multiplicative inverse: For each $P \neq O$ there exists a complex number M such that $MP = PM = U$.

(j) Distributive law: $P(Q + R) = PQ + PR$. HINT: See Problem 15 of Chapter 1.

14. Let P, Q, S, T be four complex numbers. Use the results of Problem 13 to show that:

(a) $(P + Q)^2 = P^2 + 2PQ + Q^2$.

(b) $(S + T)(S - T) = S^2 - Q^2$.

(c) $(P + Q)(S + T) = PS + QS + PT + QT$.

15. Let $E = 4e^{0i}$, $F = 4e^{\pi i}$, $G = 7e^{0i}$, and $H = 7e^{\pi i}$. Find the polar form of:

(a) EG; (b) FH; (c) FG; (d) EH.

16. Let $E, F, G,$ and H be as in Problem 15. Find the polar form of:

(a) $E + G$; (b) $F + H$; (c) $F + G$; (d) $E + H$.

17. Let P and Q both be in the set R of Problem 9 (a).

(a) Is the product PQ also in the set R? Explain.

(b) Is the sum $P + Q$ also in R? Explain.

(c) Is $Qe^{(\pi/2)i}$ in the set I of Problem 9 (b)? Explain.

18. Let $Q = 4e^{(5\pi/3)i}$ and $\overline{Q} = 4e^{(-5\pi/3)i}$. Show Q, $\overline{Q}$, $Q\overline{Q}$, and $Q + \overline{Q}$ in a diagram.

19. Let $A = 1024e^{(\pi/4)i}$ and $C = e^{(2\pi/5)i}$.

(a) Verify that $C^5 = \left(C^2\right)^5 = \left(C^3\right)^5 = \left(C^4\right)^5 = U$.

(b) Find in polar form a complex number B such that $B^5 = A$.

(c) Find in polar form and plot B, CB, C^2B, C^3B, C^4B, and C^5B.

(d) Verify that $(CB)^5 = \left(C^2B\right)^5 = \left(C^3B\right)^5 = \left(C^4B\right)^5 = A$.

20. Find in polar form and plot 5 fifth roots of $243e^{(-13\pi/18)i}$. What kind of geometrical figure has these 5 points as vertices?

21. Find in polar form and plot 7 seventh roots of $128e^{(14\pi/19)i}$.

22. Let n be an integer greater than 1 and let $D = re^{\theta i}$ be any complex number. You may assume $0 \leq \theta < 2\pi$.

(a) Find the complex number F with the smallest possible positive argument such that $F^n = U$.

(b) Find a complex number E such that $E^n = D$.

(c) Verify that the complex numbers EF^k, for $k = 0, 1, 2, ..., n - 1$ are distinct nth roots of D.

3. Negative of a Point, Subtraction, Conjugate

As in other number systems, if $P + N = O$, one writes $N = -P$ and calls N the ***negative*** of P. It follows from the definition of addition of points in the Argand Plane that N is the point such that the directed line segment $\overrightarrow{ON}$ has the same magnitude and direction as $\overrightarrow{PO}$; i.e., N is the point such that O is the midpoint of segment PN. If $P = re^{\theta i}$, then clearly

$$N = re^{(\theta+\pi)i} = re^{(\theta-\pi)i}.$$

See Figure 4.

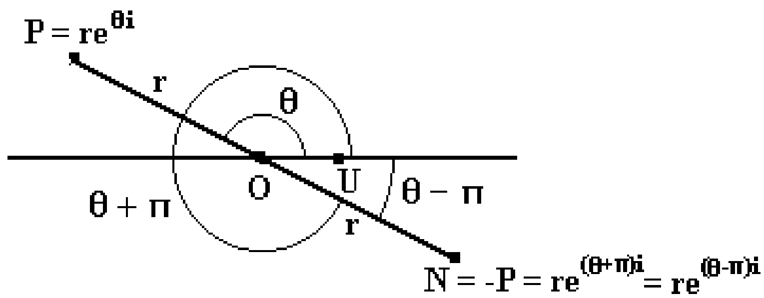

Figure 4

The ***difference*** $E - F$ is the point G such that $E = F + G$. One can also obtain the difference $G = E - F$ using the formula $G = E + (-F)$. See Figure 5.

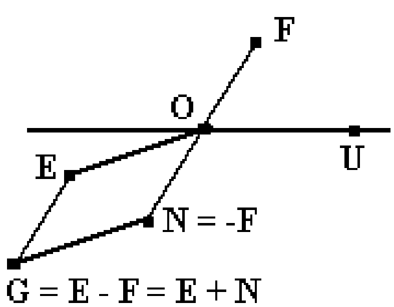

Figure 5

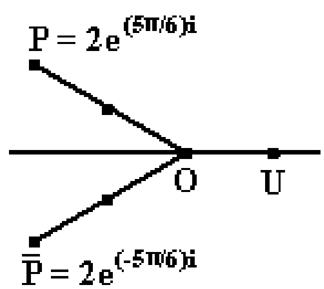

Figure 6

In the Argand Plane, the ***conjugate*** of a point $P = re^{\theta i}$ is the point $\overline{P} = re^{-\theta i}$, with the same absolute value as P but with the argument the negative of that of P. Note that a point P and its conjugate $\overline{P}$ are symmetrically situated with respect to the straight line determined by O and U. See Figure 6.

4. Reciprocal of a Point, Division

If $Q \neq O$ and $MQ = U$, one writes $M = Q^{-1}$ and this is called the ***reciprocal*** of Q. It is clear from the definition of multiplication that if $Q = re^{\theta i}$ with *r* not equal to zero, then

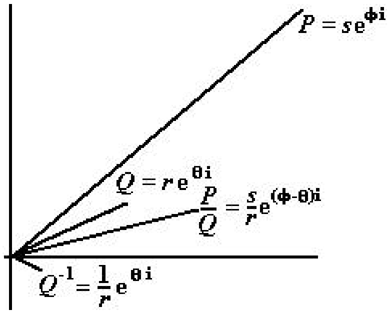

Figure 7

$Q^{-1} = \frac{1}{r}e^{-\theta i}$. The division of a complex number *P* by a complex number *Q* which is not *O* can now be defined by $P \div Q = \frac{P}{Q} = PQ^{-1}$. Thus, if $P = se^{\phi i}$, then

$$\frac{P}{Q} = PQ^{-1} = \left(se^{\phi i}\right)\left(\frac{1}{r}e^{-\theta i}\right) = \frac{s}{r}e^{(\phi-\theta)i}.$$ See Figure 7.

For example, if $P = 6e^{(2/9)\pi i}$ and $Q = 2e^{(5/36)\pi i}$ then

$$\frac{P}{Q} = \frac{6}{2}e^{\left(\frac{2}{9}-\frac{5}{36}\right)\pi i} = 3e^{(\pi/12)i}.$$

5. The Real and Imaginary Axes; Rectangular Form

Each point on the straight line through the origin *O* and the unity point *U* is expressible as $re^{\theta i}$ with $\theta = 0$ or π. The set of all these points is closed under addition and multiplication. (See Problem 17 in Exercises for Chapter 2 Sections 1 and 2) In fact, these points behave just like the real numbers under addition, subtraction, multiplication, and division. For this reason, the line through *O* and *U* is called the ***real axis*** and we identify each real number with a point on the real axis in the following manner.

The real number zero is identified with the origin *O*. A positive (real) number *r* is identified with the point re^{0i}. The material in Section 3 of this chapter makes it natural for its negative -*r* to represent the point $re^{\pi i}$. In particular, we have $U = e^{0i} = 1$.

The line perpendicular to the real axis at the origin is called the ***imaginary axis***. The imaginary unit point $e^{(\pi/2)i}$ is designated as i. Then the points $re^{(\pi/2)i}$ may be written as *r*i and points $re^{(3\pi/2)i}$ as -*r*i. An important fact is that $i^2 = e^{\pi i/2}e^{\pi i/2} = e^{\pi i} = -1$; that is, $i^2 = -1$.

Now every point on the real axis is represented by a real number *a* and every point on the imaginary axis by a ***pure imaginary number***, i.e., a number *b*i with *b* real. Note that *a* and *b* may be positive, zero, or negative. See Figure 8 below.

Let *P* be any point in the Argand Plane. Then the foot of the perpendicular from *P* to the real axis has a representation as some real number *a*. Similarly, the foot of the perpendicular from *P* to the imaginary axis is a pure imaginary number *b*i. The rule for adding points shows that

$P = a + b\mathrm{i}$. [The parallelogram with vertices 0, a, P, $b\mathrm{i}$ turns out to be a rectangle in this case. See Figure 9.]

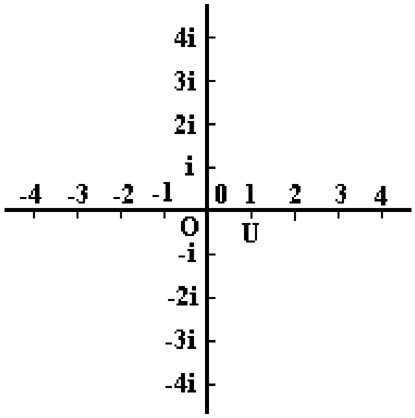

Figure 8

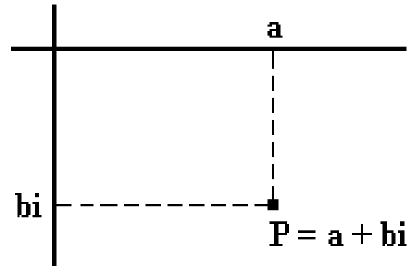

Figure 9

The representation $a + b\mathrm{i}$, with a and b real, is called the ***rectangular form*** of a complex number. The real number a is called the ***real part*** and the real number b is called the ***imaginary part*** of the complex number.

It can be shown (see Problem 17 in the exercises for this section) that in rectangular form the complex numbers have the following rules:

ADDITION $\quad (a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}.$

SUBTRACTION $\quad (a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}.$

MULTIPLICATION $\quad (a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}.$

Example 1. Convert the polar form $P = 5\sqrt{2}\,\mathrm{e}^{(3\pi/4)\mathrm{i}}$ to rectangular form $a + b\mathrm{i}$.

Solution: Let H and V be the feet of the perpendiculars from P to the real axis and the imaginary axis, respectively. We see that ΔPHO is a 45°, 45°, 90° triangle. Hence the lengths of its sides are in the ratio $1{:}1{:}\sqrt{2}$. Since the hypotenuse has length $5\sqrt{2}$, the two equal sides must have length 5. Then $H = 5\mathrm{e}^{\pi\mathrm{i}} = -5$ and $V = 5\mathrm{e}^{(\pi/2)\mathrm{i}} = 5\mathrm{i}$. Hence $P = H + V = -5 + 5\mathrm{i}$. See Figure 10.

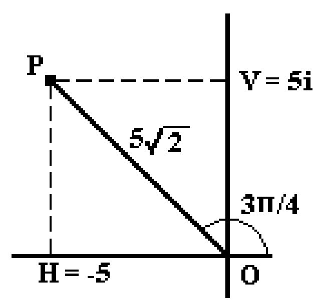

Figure 10

Example 2. Convert the rectangular form $Q = -7 - 7\sqrt{3}i$ to polar form.

Solution: Let C = -7 and $D = -7\sqrt{3}i$. [See Figure 11.] Clearly ΔQDO is a right triangle. Since the ratio of the length of side *DO* to the length of the side *QD* is $\sqrt{3}$, it is a 30°, 60°, 90° triangle and the hypotenuse *OQ* has twice the length of the shortest side *QD*, that is, the hypotenuse has length 14. Also, the counterclockwise angle from ray *OU* to ray *OQ* is 240° i.e., $4\pi/3$. Hence $Q = 14e^{(4\pi/3)i}$.

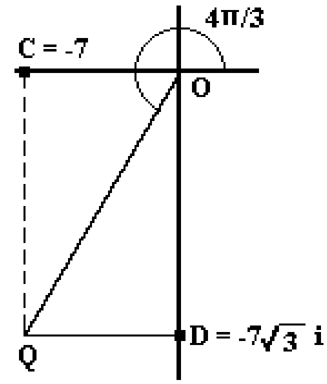

Figure 11

Example 3. Find the rectangular form of $P = 2\sqrt{2}e^{(5\pi/12)i}$.

Solution: First we note that $5\pi/12$ radians equals 75°, or 30° plus 45°. Using a 30°, 60°, 90° triangle and a 45°, 45°, 90° triangle [see Figure 12], one finds that

$$\sqrt{2}e^{(\pi/4)i} = 1 + i,$$

$$2e^{(\pi/6)i} = \sqrt{3} + i.$$

Multiplying these two complex numbers, one has

$$\sqrt{2}e^{(\pi/4)i}2e^{(\pi/6)i} = (1 + i)(\sqrt{3} + i)$$

$$2\sqrt{2}e^{(5\pi/12)i} = (\sqrt{3} - 1) + (\sqrt{3} + 1)i.$$

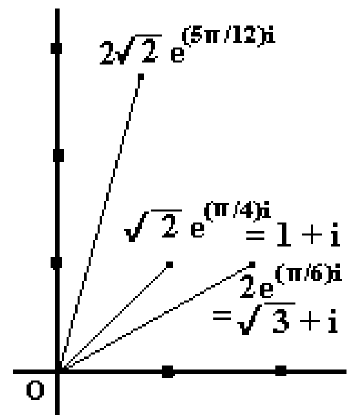

Figure 12

Exercises for Chapter II Sections 3, 4, and 5

Do NOT use a calculator for any of these problems, Give only exact answers. (See Preface)

1. Convert the following polar forms to rectangular form *a* + *b*i.

   (a) $8e^{(\pi/6)i}$; (b) $8e^{(5\pi/6)i}$; (c) $\sqrt{2}e^{(5\pi/4)i}$; (d) $9e^{(3\pi/2)i}$; (e) $4e^{(-\pi/4)i}$.

2. Convert the following rectangular forms to polar form $re^{\theta i}$. Give angles in radians.

   (a) 1 + i; (b) 7i; (c) $-5\sqrt{3} + 5i$; (d) -3; (e) -4 - 4i.

3. Find the rectangular form $a + b\mathrm{i}$ of $e^{\theta i}$ for

$$\theta = 0,\ \pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6,\ \text{and}\ \pi.$$

4. Do as in Problem 3 for $\theta = 7\pi/6,\ 5\pi/4,\ 4\pi/3,\ 3\pi/2,\ 5\pi/3,\ 7\pi/4,\ 11\pi/6,$ and 2π.

5. Find the rectangular form for the conjugate $\overline{P}$ of $P = -7 + 8\mathrm{i}$.

6. Find the rectangular form for the conjugate $\overline{P} = re^{-\theta i}$ of $P = re^{\theta i} = a + b\mathrm{i}$, where a and b are real numbers.

7. Let $S = 13e^{\phi i} = -12 + 5\mathrm{i}$. Find both the polar and the rectangular form of the point symmetric to S with respect to:

(a) the real axis; (b) the imaginary axis; (c) the origin;

(d) the straight line through O and $1 + \mathrm{i}$.

8. Do the same as in Problem 7 for $T = re^{\theta i} = h + k\mathrm{i}$ instead of S.

9. Let $A = 5e^{\alpha i} = 3 + 4\mathrm{i}$ and $B = 6\sqrt{2}e^{(3\pi/4)i} = -6 + 6\mathrm{i}$. Find in both polar and rectangular form:

(a) $-A$; (b) $-B$; (c) $\overline{A}$; (d) $\overline{B}$; (e) AB; (f) $A\overline{B}$.

10. For A and B of Problem 9, give the rectangular form of $A - B$ and plot O, A, $A - B$, and $-B$. What kind of quadrilateral are these four points the vertices of?

11. Let $A = 5e^{\alpha i} = 3 + 4\mathrm{i}$ and $D = 13e^{\beta i} = 12 - 5\mathrm{i}$. Find the rectangular form of :

(a) $65e^{(\alpha+\beta)i}$; (b) $65e^{(\alpha-\beta)i}$.

12. Let $B = 6\sqrt{2}e^{(3\pi/4)i} = -6 + 6\mathrm{i}$ and $C = 2e^{(\pi/6)i} = \sqrt{3} + \mathrm{i}$. Find $12\sqrt{2}e^{(7\pi/12)i}$ in rectangular form.

13. Given that $re^{\theta i} = 7 + 5\mathrm{i}$, use the Pythagorean Theorem to find r.

14. Given that $8e^{\phi i} = 5 + b\mathrm{i}$ and $b < 0$, find b.

15. Explain why $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$ for all real a and b and find $|11 - 8\mathrm{i}|$.

16. Use the results of Problem 16 in Chapter 1 to find complex numbers in polar and rectangular form with magnitude 1 and arguments equal to those listed below but converted to radians.

(a) 105^o; (b) 15^o; (c) 225^o; (d) 135^o; (e) 195^o.

17. Verify the rules for addition, subtraction, and multiplication of complex numbers given on page 17.

18. Let $P = re^{\theta i} = a + bi$ and $Q = se^{\phi i} = c + di$. Show that :

(a) $\overline{P + Q} = \overline{P} + \overline{Q}$; (b) $\overline{PQ} = \overline{P} \cdot \overline{Q}$.

19. Let $A = 5e^{\alpha i} = 3 - 4i$. Find in both polar and rectangular form:

(a) iA; (b) $-A$; (c) $-iA$; (d) $\overline{A}$; (e) $i\overline{A}$; (f) A^2; (g) A^3.

20. Let $B = 29e^{\beta i} = -21 + 20i$. Find the polar form (in terms of β) for :

(a) -20 - 21i; (b) 21 - 20i; (c) 20 + 21i; (d) -21 - 20i; (e) 20 - 21i.

21. Let $P = 5 - 5\sqrt{3}i$. Find P, $\overline{P}$, $-P$, and $-\overline{P}$ in both polar and rectangular form.

22. Let $Q = 2\sqrt{2}e^{(3\pi/4)i}$. Find Q, $\overline{Q}$, $-Q$, and $-\overline{Q}$ in both polar and rectangular form.

23. Given that $P = 13e^{\theta i} = 12 - 5i$ and $Q = 5e^{\phi i} = 4 + 3i$, find in rectangular form:

(a) $\overline{P} = 13e^{-\theta i}$; (b) $Q\overline{P} = 65e^{(\phi - \theta)i}$; (c) $-\overline{P} = 13e^{(\pi - \theta)i}$;

(d) $Q^2 = 25e^{2\phi i}$; (e) $PQ = 65e^{(\theta + \phi)i}$.

24. Let $A = 25e^{\alpha i} = 7 + 24i$ and $B = \sqrt{13}e^{\beta i} = 2 - 3i$. For each of the following angles θ, find r, a, and b such that $re^{\theta i} = a + bi$:

(a) $\theta = \alpha - \beta$; (b) $\theta = 2\beta$; (c) $\theta = \alpha - \pi$; (d) $\theta = \alpha + (3\pi/2)$.

25. Express $(\sqrt{3} + i)^{10}(2 - 2\sqrt{3}i)^9$ in $a + bi$ form by first converting $\sqrt{3} + i$ and $2 - 2\sqrt{3}i$ to polar form, raising to powers and multiplying in polar form, and finally converting back.

26. Do as in Problem 25 for $(5 + 5i)^{11}(-7i)^8$.

27. Let $P = 13e^{\theta i} = 5 + 12i$, $Q = 13$, and $S = P + Q$.

(a) Explain why O, Q, S, P are the vertices of a rhombus (i.e., a parallelogram with all four sides equal).

(b) Explain why $\triangle QOS = \frac{1}{2}\triangle QOP$.

(c) Find the absolute value s of S and then find $R = \left(\frac{\sqrt{13}}{s}\right)S$ in $a + bi$ form.

(d) Let R be as in (c). Explain why R and $-R$ are the 2 square roots of P.

28. Let $P = re^{\theta i} = a + bi$, $Q = r$, $S = P + Q$, and $s = |S|$. Explain why each of the following is true.

(a) O, Q, S, and P are the vertices of a rhombus.

(b) $\triangle QOS = \frac{1}{2}\triangle QOP$.

(c) $r^2 = a^2 + b^2$, $s^2 = 2r^2 + 2ra$, and $s = \sqrt{r}\sqrt{2r + 2a}$.

(d) The square roots of P are $\pm\left(\frac{\sqrt{r}}{s}\right)S = \pm\frac{S}{\sqrt{2r + 2a}} = \pm\frac{P + r}{\sqrt{2r + 2a}}$.

29. Let $P = re^{\theta i} = a + bi$ with $b > 0$. Use Problem 28 to show that the 2 square roots of P are $\pm\left(\sqrt{\frac{r + a}{2}} + i\sqrt{\frac{r - a}{2}}\right)$.

30. Let $P = re^{\theta i} = a + bi$ with $b < 0$. Use Problem 28 to show that the 2 square roots of P are $\pm\left(\sqrt{\frac{r + a}{2}} - i\sqrt{\frac{r - a}{2}}\right)$.

31. Let $V = e^{\theta i} = c + si$. Show that the square roots $e^{\theta i/2}$ and $e^{(\theta+2\pi)i/2}$ of V are

$$\pm\sqrt{\frac{1+c}{2}} \pm i\sqrt{\frac{1-c}{2}}$$

where like signs are used if $s > 0$ and unlike signs if $s < 0$.

32. Use Problem 31 to find $e^{(\pi/8)i}$ in $a + bi$ form.

33. Let $D = B - A$. See figure 13.

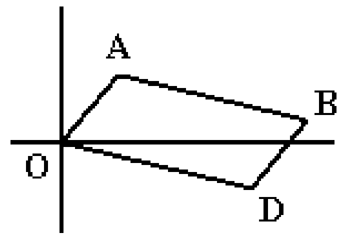

Figure 13

(a) Explain why $\vec{OD}$ and $\vec{AB}$ have the same magnitude and hence $|B - A|$ equals the distance between A and B.

(b) If $A = u + vi$ and $B = x + yi$, show that

$$|B - A| = \sqrt{(x - u)^2 + (y - v)^2}.$$

34. Let $A = 5 + 2i$, $B = 9 + 5i$, and $C = 5 + 5i$.

(a) Explain why $\angle ACB$ is a right angle.

(b) Find sides AC and CB of ΔABC and then use the Pythagorean Theorem to find side AB (i.e., the distance between A and B.)

(c) Find $|B - A|$ and $|B| - |A|$ and tell which equals the distance between A and B.

35. Sketch and identify the locus of all points A in the Argand Plane such that $|A - i| = 3$. (That is, give the graph in the Argand Plane of the equation $|A - i| = 3$.)

36. Do as in Problem 35 for each of the following equations.

(a) $|A - 4| + |A - 3i| = 5$.

(b) $|A - 4| - |A - 3i| = 0$.

37. Let $P = re^{\theta i} = a + bi$ and $Q = se^{\theta i} = c + di$ be two complex numbers with the same argument θ, $\theta \neq 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \cdots$. Also let $r > 0$ and $s > 0$. Explain why each of the following is true:

$$\frac{a}{r} = \frac{c}{s}, \qquad \frac{b}{r} = \frac{d}{s}, \qquad \frac{b}{a} = \frac{d}{c}.$$

38. Perform each of the following divisions in polar form.

(a) $12e^{(7/4)\pi i} \div 4e^{(5/8)\pi i}$; (b) $24e^{(\pi/4)i} \div 6e^{(\pi/3)i}$; (c) $15e^{(3/4)\pi i} \div 12e^{(-\pi/3)i}$.

39. Let P and Q be complex numbers with $|Q| = s > 0$. Show that $\dfrac{P}{Q} = \dfrac{1}{s^2} P\overline{Q}.$

40. Use the result of Problem 39 to perform each of the following divisions.

(a) $\dfrac{10+11i}{3+2i}$; (b) $\dfrac{9-38i}{4-3i}$; (c) $\dfrac{11+3i}{2+2i}$; (d) $\dfrac{29+31i}{2-7i}$.

41. Show that if $c + di \neq 0$, then $\dfrac{a+bi}{c+di} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2} i.$

42. Let A = 6 and B = 4 + 3i.

(a) Plot the points A and B and complete the ΔOAB.

(b) Find the rectangular form of $A' = Ae^{(2/3)\pi i}$ and $B' = Be^{(2/3)\pi i}$.

(c) On the same graph you used for part (a), plot the points A' and B' and complete the $\Delta OA'B'$. How would you describe the effect that multiplying by $e^{(2/3)\pi i}$ had on the ΔOAB?

43. Let A = 5 - i and B = 5 + i. Rotate ΔOAB 30° counterclockwise about the origin. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

44. Let A = 2 + 5i and B = -1 + 4i. Rotate ΔOAB 135° clockwise about the origin. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

45. Let A = -1 - 2i and B = 3 - 2i. Rotate ΔOAB 60° counterclockwise about the origin and at the same time stretch it so that each side is twice the length of the original. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

46. Let A = 6 + 12i and B = -3 + 15i. Rotate ΔOAB 120° clockwise about the origin and at

the same time shrink it so that each side is one third the length of the original. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

47. Let $A = 5 - 2i$ and $B = 3 + 2i$. Rotate ΔOAB 45º counterclockwise about the origin and at the same time stretch it so that the area is twice the area of the original. Find the new coordinates of the vertices and graph the triangle before and after the rotation.

48. Let $A = 4 + 3i$ and $B = 2 + 5i$. Rotate ΔOAB counterclockwise and stretch it until vertex A is at -5 + 15i. Find the new coordinate of B and graph the triangle before and after the rotation.

6. Complex Numbers on the Calculator - Polar Form

Our first job is to get the calculator options set up for the task at hand. Key MODE DA DA and F2-CHOOS to see a drop down list with three choices of angle measure. Use DA or UA as needed to select Degrees from the list then press F6-OK. This puts the calculator in degree mode. Now key DA and F2-CHOOS to see three choices for coordinate systems. Again use DA or UA as needed to select Polar from this list then press F6-OK. For more information about angle modes and coordinate systems see page 1-22 of ***UG***. To make sure that the constants in the calculator give numeric rather than symbolic results. Key F1-FLAGS then use DA and F3-CHK as needed to make sure that Flags 2 and 3 are checked and Flag 27 is clear, then F6-OK. For more information about the roll of Flag 2 see page 2-64 of ***UG***. Finally press F3-CAS DA DA to highligt "_Approx." The other two items on that row are "_Numeric" on the left and "_Complex" on the right. Use LA, RA, and F3-CHK as needed to check all three of these items, then press F6-OK on this dialog box and on the next to return to the main screen.

Now look at the row of annunciators at the top of the screen. They should be "Degree," "R∠Z," "HEX," "C~." and "'X'."

In the calculator, the complex number $P = re^{\theta i}$ is expressed as $(r, \angle\theta)$. One limitation of the calculator is that it will not accept variables for the magnitude and argument, only numbers. In particular, if $P = 3e^{(\pi/4)i}$, it can't be expressed as $(3, \angle\frac{\pi}{4})$, it must be expressed as $(3, \angle.785398163398)$. As a general rule, this limitation makes working with degrees much easier than with radians. We will, therefore, start with degrees, but we will see some tricks that will make working with radians not quite as bad as it looks. Actually, working in degrees has its own much more subtle problems. Since the calculator actually works with radians internally, there are two conversions in moving data from the keyboard to the display, and these sometimes cause annoying roundoff errors. To avoid this, set your display to Fix 2 for the time being.

NOTE: The angle mark, $\angle$, is created with the key stroke sequence AS RS 6.

Calculator Example 2.6.1

Let $P = (2,\angle 43^o)$ and $Q = (3,\angle 29^o)$, find PQ. With the calculator in degrees and polar coordinates, key in LS ( ) 2 (NOTE: At this point you may key in the comma which takes two key strokes or a space, SPC, and the calculator will convert it to a comma after you hit ENTER, but neither is necessary.) AS RS 6 43 ENTER LS ( ) 3 AS RS 6 29 ×. You should see $(6.00,\angle 72.00)$ on the display. This clearly satisfies the definition of product from Section 2 of this chapter.

Now try it with $P = (3.4,\angle 87^o)$ and $Q = (6.3,\angle 112^o)$. The result you get is $(21.42,\angle -161.00)$. The magnitude is certainly correct, but we were expecting an argument of 199. What happens is that the calculator always normalizes the argument so that $-180^o < \theta^o \le 180^o$, or in radians, $-\pi < \theta \le \pi$.

Calculator Example 2.6.2

Let $P = (4,\angle 30^o)$ and $Q = (2,\angle 150^o)$ and find $P + Q$. Key in the complex numbers as in the previous example, and press + to find the sum. The result is $(3.46,\angle 60.00)$ to two decimal places. We now use the definition of sum in Section 2 and geometry to verify that this answer is correct. See Figure 14.

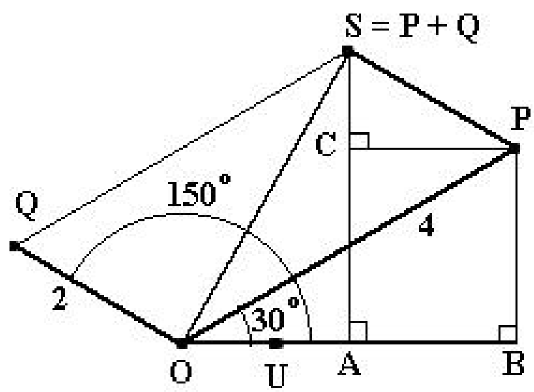

Figure 14

Construct SA and PB perpendicular to OU, and PC parallel to OU. Now ΔOBP is a 30°, 60°, 90° triangle with hypotenuse 4, so $PB = 2 = CA$, and $OB = 2\sqrt{3}$. Since $\angle CPS = 180^o - 150^o = 30^o$, ΔCPS is a 30°, 60°, 90° triangle with hypotenuse 2, so $CP = \sqrt{3} = AB$ and $SC = 1$. $OA = OB - AB = \sqrt{3}$ and $AS = AC + SC = 3$, so ΔOAS is a 30°, 60°, 90° triangle. Thus $S = (2\sqrt{3},\angle 60^o)$, which agrees with our calculator answer at least to the level of accuracy possible on the calculator.

Calculator Example 2.6.3

An engineer wishes to build a straight railroad from town A to town B with a tunnel through the mountain from point C to point D. [See Figure 15] She cannot, of course see B from A, so she does not know what direction to go from A to find one end of the tunnel at C, nor what direction to go from B to find the other end of the tunnel at D. She can, however, see a tall tree at P from A, from P she can see a large rock at Q, and from Q she can see B. How should she proceed?

Solution: She visualizes the map as the Argand Plane with the origin at *A* and *U* one kilometer east of *A*. She finds that from *A* to *P* is 10.13 kilometers in a direction 19.8° measured counterclockwise from east, hence $\vec{AP}$ can be thought of as the complex number $(10.13,\angle 19.8)$.

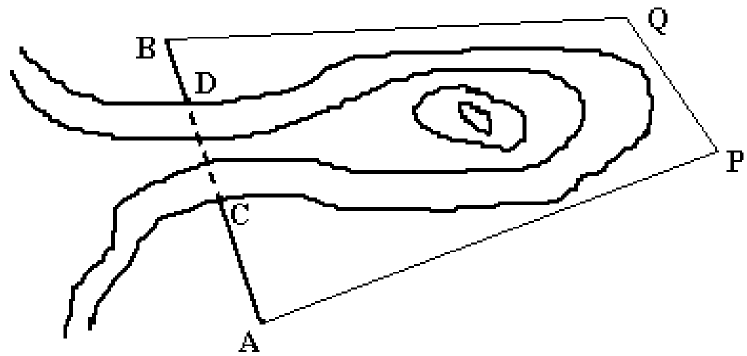

Figure 15

From *P* to *Q* is 125.3° measured counterclockwise from east and the distance is 3.17 kilometers, so $\vec{PQ}$ can be thought of as the complex number $(3.17,\angle 125.3)$. Finally, she finds that from *Q* to *B* is 8.68 kilometers and the direction is 177.4° clockwise from east, so $\vec{QB}$ is considered to be the complex number $(8.68,\angle -177.4)$. She adds these three complex numbers on her calculator and finds the sum to be $(5.71,\angle 99.8)$, so from *A* to *B* is a distance of 5.71 kilometers in the direction 99.8° counterclockwise from east. She now goes in that direction from *A* until she comes to the mountain at a point she labels *C*, then goes in the opposite direction from *B* until she comes to the mountain at the point she labels *D*.

As mentioned above, the calculator's representation of a complex number $P = (r,\angle\theta)$ requires *r* and θ to be expressed in decimal form. If $P = 2\sqrt{3}e^{\frac{\pi}{7}i}$ one can compute the magnitude and argument to twelve significant figures on the calculator, then key them into the parentheses to get $(3.46410161514,\angle .448798950513)$, but this is a very tedious and error prone procedure. We will consider some tricks for entering such complex numbers more efficiently. For what follows, set the calculator numeric mode to standard to see the full 12 significant digits.

The easiest case is when *r* is "nice" and the argument is of the form $\theta = \frac{a}{b}\pi$, with *b* a divisor of 180. Such an argument is an integer when expressed in degrees, hence easy to enter in that form. For example, if $P = 2e^{\frac{\pi}{4}i}$, the argument is 45°. To get this complex number into the calculator in radian form, first set the calculator to degree mode, enter the number in degree form, then switch the calculator to radian mode; the 45 will change to .785398163398.

We now return to the more general case, $P = 2\sqrt{3}e^{\frac{\pi}{7}i}$. Although there are several ways to handle this, we will discuss only one; the equation writer. (See page 2-10 and Appendix E of ***UG***.) With the calculator in radian mode, key in RS EQW 2 × $\sqrt{x}$ 3 RA × LS e^x LS π ÷ 7 RA × LS i ENTER EVAL. You should now see $(3.46410161514,\angle .448798950513)$ on the display. Note that we have used the equation writer to create the algebraic object $2\sqrt{3}e^{\frac{\pi}{7}i}$, then used the evaluation command, EVAL, to convert it to the complex number (3.46410161514,∠.448798950513).

The concepts of negative, subtraction, and conjugate are all easily handled on the calculator. The +/- key converts the complex number in level 1 of the stack to its negative. Enter your favorite complex number and try it! Note that pressing +/- twice returns the original number as it should. The - key will subtract the complex number in level 1 from the complex number in level 2. Try it with your choice of complex numbers *A* and *B*. After computing *S* = *A* - *B*, verify that *S* + *B* returns your original *A*. There is also a conjugate function, but it is in one of the menus. Key in LS MTH NXT F3-CMPLX NXT and you will find CONJ as the third item in the menu. Key in a complex number and try it. Again, pressing it twice gives back the original complex number as expected.

There are two other functions on this menu page; NEG and SIGN. NEG works the same as the +/- key and SIGN converts the complex number on level 1 into a complex number with the same argument but with absolute value 1. Press NXT and you see six more functions related to complex numbers. The first four will be discussed in the next section, but ABS returns the magnitude of the complex number on level 1, and ARG returns its argument. ABS and ARG are also on the keyboard in conjunction with the ÷ key.

7. Rectangular Form on the Calculator

If the calculator is still in polar mode from the previous section, press MODE and change the Coordinate system back to Rectangular.   The complex number *a* + *b*i has rectangular form (*a*,b) on the calculator. For example, to enter 3 + 5i into the calculator key in LS ( ) 3 SPC (one could key in the comma here instead of SPC, but that would take two key strokes and the calculator will turn the space into a comma anyway) 5 ENTER. At this point we can also discuss the first four functions in the complex number menu which were mentioned in the previous section. Key LS MTH NXT F3-CMPLX to get back into the complex number menu. Assuming there is a complex number on level 1 of the stack (with the calculator in either rectangular or polar mode) F1-RE returns the real part of that complex number and F2-IM returns the imaginary part of the complex number on level 1. The function F3-C->R takes the complex number from level 1 then puts the real part on level 2 and the imaginary part on level 1 while F4-R->C reverses that process. It takes a real number from level 2 as the real part and a real number from level 1 as the imaginary part and forms a complex number which it puts on level 1.

Calculator Example 2.7.1

We will use the calculator to do Examples 1 and 2 from Section 5 of this chapter. For Example 1, we were to find the rectangular form of $P = 5\sqrt{2}e^{(3\pi/4)i}$. With the calculator set to rectangular mode and standard display, use the equation writer to key in the polar form of *P*: RS

EQW 5 × $\sqrt{x}$ 2 RA × LS e^x 3 × LS π UA ÷ 4 RA × LS i ENTER EVAL. The result will be (-4.99999999998,5). The real part should, of course, be -5, but we are seeing the unavoidable roundoff errors which occur when we use a finite device to approximate real numbers.

For Example 2 we were to find the polar form for $Q = -7 - 7\sqrt{3}i$. We will place the real part on level 2 of the stack and the imaginary part on level 1, then use the R->C command to create the complex number. Set the calculator to polar coordinates, standard display, radian angle mode, and bring up the complex number menu. Now key 7 +/- ENTER ENTER 3 $\sqrt{x}$ × F4-R->C. We should now see (14,∡−2.09439510239). There is one more thing we can try which sometimes gives us an answer in a form we would like to see it. Key F6-ARG LS CONVERT F4-REWRI NXT F6– →Q*p* and we see $-2/3*p$ on the display. The F6– →Q*p* command works to convert a decimal expression into a rational number times π if it's not to far from one of the "nice" angles. If, however, you compute $\frac{251\pi}{3163}$ as a decimal then try F6– →Q*p* to get it back, you get a very strange result.

Calculator Example 2.7.2

Find |(3 + 7i)(2 - 5i)|. We key in the problem: LS ( ) 3 SPC 7 ENTER LS ( ) 2 SPC 5 +/- × LS ABS. We see the answer 41.0121933088. Notice that it doesn't matter if the calculator is in rectangular or polar coordinates when we key in the problem.

Exercises for Chapter II Sections 6 and 7

For problems 1 - 8 give the magnitudes, real parts, and imaginary parts to two decimal places and the arguments in radians to four decimal places.

1. Let $C = 2e^{-3\pi i/4}$, $D = 4e^{\pi i/6}$, $E = e^{2\pi i}$, and $F = e^{-2\pi i/3}$. Give the polar form (i.e., $re^{\theta i}$ form) for each of the following products. Compare your answers to the exact values obtained in Problem 1 of Exercises for Chapter 2 Sections 1 and 2.

   (a) *CD*; (b) *CE*; (c) *CF*; (d) C^2.

2. Let $A = 1024e^{(\pi/4)i}$ and $C = e^{(2\pi/5)i}$.

   (a) Use the y^x key to find a $B = A^{1/5}$ in polar form.

   (b) Find in polar form CB, C^2B, C^3B, C^4B, and C^5B.

   (c) Compare your answers with the exact values obtained in Problem 19 of Exercises for

Chapter 2 Sections 1 and 2.

3. Find the polar form for 5 fifth roots of $249.47e^{0.7493i}$.

4. Find the polar form for 7 seventh roots of $8274.85e^{-0.4444i}$.

5. Convert the following polar forms to rectangular form $a + b$i.

(a) $8e^{(\pi/6)i}$; (b) $8e^{(5\pi/6)i}$; (c) $\sqrt{2}e^{(5\pi/4)i}$; (d) $9e^{(3\pi/2)i}$; (e) $4e^{(-\pi/4)i}$.

(f) Compare your answers with the exact values obtained in Problem 1 of Exercises for Chapter 2 Sections 3, 4, and 5.

6. Convert the following rectangular forms to polar form $re^{\theta i}$.

(a) 1 + i; (b) 7i; (c) $-5\sqrt{3} + 5i$; (d) -3; (e) -4 - 4i.

(f) Compare your answers with the exact values obtained in Problem 2 of Exercises for Chapter 2 Sections 3, 4, and 5.

7. Find complex numbers in polar and rectangular form with magnitude 1 and arguements equal to those listed below but converted to radians.

(a) 105^o; (b) 15^o; (c) 225^o; (d) 135^o; (e) 195^o.

(f) Compare your answers with the exact values obtained in Problem 16 of Exercises for Chapter 2 Sections 3, 4, and 5.

8. Find the rectangular form of $e^{(\pi/8)i}$ by entering $e^{(\pi/4)i}$ then pressing the square root key. Compare your answers with the exact values optained in Problem 32 of Exercises for Chapter 2 Sections 3, 4, and 5.

For problems 9 - 12 let $A = 3.47 - 6.52i$, $B = 4.32e^{1.1275i}$, $C = -8.46 + 3.92i$ and $D = 0.39e^{-2.0727i}$. Evaluate the given expressions and leave your answers in both polar and rectangular form. Give the magnitudes, real parts, and imaginary parts to two decimal places and the arguments in radians to four decimal places. HINT: It may be helpful to store the values of *A*, *B*, *C*, and *D* in your calculator memory.

9. (a) $(A + B - C)D$; (b) $\dfrac{A+B}{C-D}$; (c) $(A^2 - C^2)D$; (d) $(A + C)^3D$.

10. (a) $(A + B)(C - D)$; (b) $\frac{A - C}{B + D}$; (c) $(A^3 + C^3)D$ (d) $(A - C)^2D$.

11. Find the distance between A and B.

12. Find the distance between C and D.

In navigation directions are called bearings and are measured in degrees clockwise from north. Thus, if you are traveling east, you are traveling on a bearing of 90^o; and if you are traveling south west, you are traveling on a bearing of 225^o. For problems 13 and 14 give directions as bearings to the nearest degree and distances as miles to the nearest tenth of a mile.

13. A Coast Guard patrol boat leave the Coast Guard station and travels 13.1 miles on a bearing of 37^o. It then turns to a bearing of 106^o and travels 19.1 miles. From there it travels 13.2 miles on a bearing of 250^o at which time it receives a distress call from a fishing boat. To reach the fishing boat it travels 11.4 miles on a bearing of 348^o.

    (a) Covert all of the bearings above to angles measured counterclockwise from east.

    (b) There is an injured crewman on the fishing boat and the Coast Guard medic decides he should be helicoptered to a hospital. What distance and bearing should the helicopter fly from the Coast Guard station to the fishing boat? HINT: See Calculator Example 2.6.3.

    (c) It is known that the nearest hospital is 19.5 miles on a bearing of 166^o from the Coast Guard station. What distance and bearing must the helicopter fly from the fishing boat to the hospital?

14. A geological team leaves their base camp in a jeep and travel across the desert for 3.1 miles on a bearing of 27^o. From there they travel for 6.7 miles on a bearing of 257^o. They leave that location on a bearing of 146^o, but their jeep breaks down after 3.2 miles. They radio the base camp for a second jeep to come pick them up.

    (a) What direction and distance must the second jeep travel from the base camp to reach the stranded team?

    (b) It is known that the nearest repair facility is 6.4 miles on a bearing of 229^o from the base camp. If the decision is made to tow the disabled jeep directly to the repair facility, what direction and distance should they travel from the site of the breakdown?