Return to List of Lessons

Mathematica @
Thiel College

LESSON 4: Eigenvalues, Eigenvectors & Diagonalization

Let A be a square matrix. We call λ an eigenvalue of the matrix A if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero vector **x** and **x** is called an eigenvector corresponding to the eigenvalue λ. In this lesson, we will learn how to find eigenvalues and eigenvectors of a square matrix using Mathematica. We will also study whether A is diagonalizable by finding the number of linearly independent eigenvectors of A. Note that A is diagonalizable if and only if $A = PDP^{-1}$ for a diagonal matrix D and an invertible matrix P. In other words, A is diagonalizable if and only if A is similar to a diagonal matrix.

Example 1: Let $A = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$. Find the eigenvalues and linearly independent eigenvectors of A. Determine if A is diagonalizble.

Step 1. Define a square matrix in Mathematica.

Type **A = {{6, −2, 0}, {−2, 9, 0}, {5, 8, 3}}**, then **Shift + Enter**

The result: {{6, −2,0}, {−2,9,0}, {5,8,3}}

So now A represents the matrix above.

Step 2. Find eigenvalues of A. Use '**Eigenvalues[A]**' to find eigenvalules of A.

Type **Eigenvalues[A]**, then **Shift + Enter**

The result: {10,5,3}

The result means that there are three distinct eigenvalues 10, 5, and 3 of A. Note that these three distinct eigenvalues of a 3 × 3 matrix A guarantee that A is diagonalizable. We will find a diagonal matrix which is similar to A in **Step 3**.

Step 3. Find independent eigenvectors of A. Use '**Eigenvectors[A]**' to find independent eigenvectors of A

Type **Eigenvectors[A]**, then **Shift + Enter**

The result: $\{\{-7,14,11\},\{2,1,9\},\{0,0,1\}\}$

These three eigenvectors in the result should be interpreted as column vectors such as

$$\begin{bmatrix}-7\\14\\11\end{bmatrix}, \begin{bmatrix}2\\1\\9\end{bmatrix}, \text{and } \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

An eigenvalue is not specified for each eigenvector in the result. The result in **Step 3** should be understood as follows: The first eigenvector $\begin{bmatrix}-7\\14\\11\end{bmatrix}$ is an eigenvector corresponding to the first eigenvalue 10 in the result of **Step 2**. So $\begin{bmatrix}2\\1\\9\end{bmatrix}$ is an eigenvector corresponding to the second eigenvalue 5 and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is an eigenvector corresponding to the last eigenvalue 3. It is obvious that these three eigenvectors are linearly independent since they are eigenvectors corresponding to distinct eigenvalues. From **Step 2** and **Step 3**, we can derive the following relationship which implies that A is diagonalizable,

$$\mathrm{A} = \begin{bmatrix}-7 & 2 & 0\\14 & 1 & 0\\11 & 9 & 1\end{bmatrix}\begin{bmatrix}10 & 0 & 0\\0 & 5 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}-7 & 2 & 0\\14 & 1 & 0\\11 & 9 & 1\end{bmatrix}^{-1}.$$

<u>Example 2</u> : Diagonalize the following matrix, if possible.

$$\mathrm{A} = \begin{bmatrix}2 & 4 & 3\\-4 & -6 & -3\\3 & 3 & 1\end{bmatrix}.$$

Step 1. Define the matrix.

Type $\mathbf{A = \{\{2,4,3\},\{-4,-6,-3\},\{3,3,1\}\}}$, then **Shift + Enter**

The result: $\{\{2,4,3\},\{-4,-6,-3\},\{3,3,1\}\}$

<u>Step 2</u>. Find eigenvalues.

Type $\mathbf{Eigenvalues}[\mathbf{A}]$, then **Shift + Enter**

The result: $\{-2,-2,1\}$

The result says that there are two distinct eigenvalues -2 and 1. The multiplicity of the eigenvalue -2 is two since -2 is appeared twice in the result.

Step 3. Find eigenvectors.

Type **Eigenvectors[A]**, then **Shift + Enter**

The result: $\{\{-1,1,0\},\{0,0,0\},\{1,-1,1\}\}$

So there are two linearly independent eigenvectors $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ which are eigenvectors corresponding to eigenvalues -2 and 1 respectively. Note that the second vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is not an eigenvector corresponding to the eigenvalue -2 since the zero vector cannot be an eigenvector. So there are two linearly independent eigenvectors of a 3×3 matrix A. Therefore A is **not** diagonalizable.

In Example 2, we found that a 3×3 matrix A has two distinct eigenvalues and it is not diagonalizable. But it is possible for an $n \times n$ matrix which has less than n distinct eigenvalues to be diagonalizable. It will be shown in the next example.

Example 3 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix}1 & 3 & 3\\-3 & -5 & -3\\3 & 3 & 1\end{bmatrix}$$

Step 1. Define the matrix.

Type $\mathbf{A = \{\{1,3,3\},\{-3,-5,-3\},\{3,3,1\}\}}$, then **Shift + Enter**

The result: $\{\{1,3,3\},\{-3,-5,-3\},\{3,3,1\}\}$

Step2. Find eigenvalues of A.

Type **Eigenvalues[A]**, then **Shift + Enter**

The result: $\{-2,-2,1\}$

So there are two distinct eigenvalues -2 and 1 of A. Note that the multiplicity of -2 is two.

Step 3. Find eigenvectors of A.

Type **Eigenvectors[A]**, then **Shift + Enter**

The result: $\{\{-1,0,1\},\{-1,1,0\},\{1,-1,1\}\}$

The result in Step 3 implies that A is **diagonalizable** since there are **<u>three</u>** linearly independent eigenvectors of a 3 × 3 matrix A. Combining the results in Step 2 and Step 3 we have

$$\mathrm{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$$

Return to List of Lessons