Chapter III

Trigonometry Using Complex Numbers

1. The Trigonometric Functions

Let θ be any real number. To obtain the trigonometric functions of θ one finds $r > 0$, a, and b such that $re^{\theta i} = a + bi$ and then uses the definitions:

$$\text{(D)}\qquad \begin{aligned} \cos(\theta) &= \frac{a}{r}, & \sin(\theta) &= \frac{b}{r}, & \tan(\theta) &= \frac{b}{a}, \\ \sec(\theta) &= \frac{r}{a}, & \csc(\theta) &= \frac{r}{b}, & \cot(\theta) &= \frac{a}{b}. \end{aligned}$$

The above three letter functions are abbreviations for cosine, sine, tangent, secant, cosecant, and cotangent, respectively. We note that the requirement $r > 0$ insures that cosine and sine are defined for all real numbers θ, but the other four functions will be undefined for values of θ which cause a zero in the denominator. [See Problem 4 below.] When any of these functions is defined, however, Problem 37, Exercises for Chapter 2 Sections 4, 5, and 6 shows that the definition is not ambiguous; that is, the functions are ***well defined***. For example, since $\sqrt{2}e^{(3\pi/4)i} = -1 + i$, we have

$$\cos(3\pi/4) = \frac{a}{r} = \frac{-1}{\sqrt{2}},\ \sin(3\pi/4) = \frac{b}{r} = \frac{1}{\sqrt{2}},\ \tan(3\pi/4) = \frac{b}{a} = \frac{1}{-1} = -1,$$

and the other three functions are the reciprocals of these. The same results would be obtained from $5\sqrt{2}e^{(3\pi/4)i} = -5 + 5i$ or any other nonzero complex number with $3\pi/4$ as argument.

The definitions show that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)},\ \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)},\ \sec(\theta) = \frac{1}{\cos(\theta)},\ \text{and}\ \csc(\theta) = \frac{1}{\sin(\theta)}.$$

The definitions also imply that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Then

$$re^{\theta i} = a + bi = r\cos(\theta) + r\sin(\theta)i = r\big(\cos(\theta) + i\sin(\theta)\big).$$

For complex numbers of absolute value 1, i.e., for $r = 1$, this becomes the ***Euler Formula***

$$e^{\theta i} = \cos(\theta) + i\sin(\theta).$$

Replacing θ with $-\theta$, we get $e^{-\theta i} = \cos(-\theta) + i\sin(-\theta)$. Taking conjugates of each side of the Euler Formula, we get $e^{-\theta i} = \cos(\theta) - i\sin(\theta)$. These equations show that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Example 1. ***Double angle formulas*** for cosine and sine.

We use the Euler Formula to obtain $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ as follows:

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= e^{2\theta i} = \left(e^{\theta i}\right)^2 \\ &= \left(\cos(\theta) + i\sin(\theta)\right)^2 \\ &= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\left(2\sin(\theta)\cdot\cos(\theta)\right).\end{aligned}$$

Equating the real and imaginary parts on each side of the equation, we have

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

and

$$\sin(2\theta) = 2\sin(\theta)\cdot\cos(\theta).$$

These are the double angle formulas for cosine and sine respectively.

Similarly one can derive half angle formulas for $\cos(\theta/2)$ and $\sin(\theta/2)$ in terms of $\cos(\theta)$ using the fact that $e^{(\theta/2)i}$ is one of the two square roots of $e^{\theta i}$. [See Problems 6 and 14 below.]

Example 2. ***Addition Formulas*** for cosine and sine.

We use the Euler Formula to express $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ in terms of $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\beta)$, and $\sin(\beta)$ as follows:

$$\begin{aligned}e^{(\alpha + \beta)i} = e^{\alpha i}e^{\beta i} &= \left(\cos(\alpha) + i\sin(\alpha)\right)\left(\cos(\beta) + i\sin(\beta)\right) \\ \cos(\alpha + \beta) + i\sin(\alpha + \beta) &= \left(\cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)\right) \\ &\quad + i\left(\sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta)\right).\end{aligned}$$

If we now equate the real and imaginary parts on each side of the equation, we get

$$\cos(\alpha + \beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)$$

and

$$\sin(\alpha + \beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta).$$

These are the addition formulas for the cosine and the sine, respectively. A symbolic aid for remembering these formulas is

$$(C + \mathrm{i}S)(c + \mathrm{i}s) = (Cc - Ss) + \mathrm{i}(Sc + Cs).$$

Example 3. ***Subtraction formula*** for tangent.

We seek $\tan(\alpha - \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$. First we note that

$$\begin{aligned} e^{\theta i}\sec(\theta) &= \big(\cos(\theta) + i\sin(\theta)\big)\sec(\theta) \\ &= \cos(\theta)\cdot\sec(\theta) + i\sin(\theta)\cdot\sec(\theta) \\ &= \cos(\theta)\cdot\frac{1}{\cos(\theta)} + i\sin(\theta)\cdot\frac{1}{\cos(\theta)} \\ &= 1 + i\tan(\theta). \end{aligned}$$

What we need here is only that there is a real number k such that $ke^{\theta i} = 1 + i\tan(\theta)$. Taking conjugates, we have $ke^{-\theta i} = 1 - i\tan(\theta)$. Thus

$$\begin{aligned} re^{\alpha i}se^{-\beta i} &= \big(1 + i\tan(\alpha)\big)\big(1 - i\tan(\beta)\big) \\ rse^{(\alpha-\beta)i} &= \big(1 + \tan(\alpha)\cdot\tan(\beta)\big) + i\big(\tan(\alpha) - \tan(\beta)\big). \end{aligned}$$

It now follows from the definition of the tangent that

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\cdot\tan(\beta)}.$$

This is the subtraction formula for the tangent.

Do NOT use your calculator for the exercises in this section or in Section 2.

Exercises for Chapter 3 Section 1

1. Given that $\tan(\theta) = 3/5$ and $180^\circ < \theta < 270^\circ$, find:

(a) $\cos(\theta)$, (b) $\sin(\theta)$, (c) $\sec(\theta)$, (d) $\cot(\theta)$.

2. Let β be an angle with $\sin(\beta) = \dfrac{2}{\sqrt{13}}$ and $90^\circ < \beta < 180^\circ$. Find *r, a, b* such that $re^{\beta i} = a + bi$ with the given β and then find $\cos(\beta)$ and $\tan(\beta)$.

3. Let $\tan(\alpha) = 8/15$ with α acute. Find a complex number in both polar and rectangular forms having α as its argument and then find $\cos(\alpha)$ and $\sin(\alpha)$.

4. It is clear from definitions (D) that there are values of θ for which some of the trigonometric functions are not defined because the denominator of the defining fraction will be zero.

   (a) Characterize all the values of θ for which the cosecant and cotangent are not defined.

   (b) Characterize all the values of θ for which the tangent and secant are not defined.

5. Let α and β be as in Problems 2 and 3 above. Use **operations on complex numbers** and the results of Problems 2 and 3 to find the following: [Do **not** use subtraction formulas, double angle formulas, etc.]

   (a) $\cos(\alpha - \beta)$, (b) $\sin(2\beta)$, (c) $\tan\left(\dfrac{\pi}{2} - \beta\right)$, (d) $\cos(\pi - \alpha)$.

6. Given that $re^{\phi i} = -6 + 7i$, use square roots of complex numbers to find all possibilities for

   (a) $\sin(\phi/2)$, (b) $\tan(\phi/2)$, (c) $\cos(\phi/2)$.

7. Prove the ***Pythagorean Identity***; $\sin^2(\theta) + \cos^2(\theta) = 1$ for all angles θ.

8. Let $e^{\theta i} = c + is$. Find all six trigonometric functions of θ in terms of *c* and *s*.

9. Let $e^{\theta i} = c + is$. Find the six trigonometric functions of θ in terms of *c* for

   (a) $0 \le \theta \le \pi$ and (b) $\pi \le \theta \le 2\pi$. [HINT: Use $s = \pm\sqrt{1 - c^2}$ (see Problem 7) and the results of Problem 8.]

10. Let $e^{\theta i} = c + is$. Express $e^{2\theta i}$, $e^{3\theta i}$, $e^{4\theta i}$, and $e^{5\theta i}$ in terms of *c* and *s*.

11. Use Problem 10 to express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ for $n = 1, 2, 3, 4, 5$.

12. Express $\cos(n\theta)$ and $\dfrac{\sin(n\theta)}{\sin(\theta)}$ in terms of $\cos(\theta)$ for $n = 1, 2, 3, 4, 5$.

13. Derive the addition, subtraction, and double angle formulas for the cosine, sine, tangent, and cotangent using complex numbers. (Some of these are in Examples 1, 2, and 3.)

14. (a) Derive the formulas $\cos(\alpha/2) = \pm\sqrt{\dfrac{1 + \cos(\alpha)}{2}}$ and $\sin(\alpha/2) = \pm\sqrt{\dfrac{1 - \cos(\alpha)}{2}}$ using square roots of complex numbers. Explain choice of ± sign.

(b) Use Problem 28, Exercises for Chapter 2 Sections 4, 5, and 6 to show that

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)}.$$

15. (a) Complete the following table: [HINT: See Problem 3, Exercises for Chapter 2 Sections 4, 5, and 6. and use $\sin(-\theta) = -\sin(\theta)$.]

x	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(x)$					0	1/2			

(b) Explain why $\sin(x + \pi) = -\sin(x)$ and $\sin(x + 2\pi) = \sin(x)$.

(c) Use parts (a) and (b) to tabulate $y = \sin(x)$ for $\pi/2 \le x \le 3\pi/2$.

(d) Graph $y = \sin(x)$ for $-\pi \le x \le 2\pi$.

16. (a) Graph $y = \cos(x)$ for $-\pi \le x \le 2\pi$.

(b) Graph $y = \tan(x)$ for $-\pi/2 < x < \pi/2$ and $\pi/2 < x < 3\pi/2$.

17. Use the formula $\csc(x) = 1/\sin(x)$ and Problem 15 to graph $y = \csc(x)$ for $0 < x < \pi$ and $\pi < x < 2\pi$.

18. Graph:

(a) $y = \sec(x)$ for $-\pi/2 < x < \pi/2$ and $-3\pi/2 < x < -\pi/2$.

(b) $y = \cot(x)$ for $0 < x < \pi$ and $\pi < x < 2\pi$.

19. Consider a right triangle with α one of its acute angles. Let *hyp* be the length of the hypotenuse, *adj* the length of the side adjacent to α, and *opp* the length of the side opposite α. Verify each of the following:

(a) $\cos(\alpha) = \frac{adj}{hyp}$, (b) $\sin(\alpha) = \frac{opp}{hyp}$, (c) $\tan(\alpha) = \frac{opp}{adj}$,

(d) $\sec(\alpha) = \frac{hyp}{adj}$, (e) $\csc(\alpha) = \frac{hyp}{opp}$, (f) $\cot(\alpha) = \frac{adj}{opp}$.

20. Use the double angle formula for the cosine and the Pythagorean Identiy (see Problem 7) to verify each of the following identities:

a) $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$;

b) $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$.

2. The Inverse Trigonometric Functions.

If f and g are functions such that $f(a) = b$ if and only if $g(b) = a$, then f and g are ***inverse functions*** of each other. For example, the functions f and g with $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverse functions of one another since $b = a^3$ if and only if $a = \sqrt[3]{b}$. [Note that the inverse function of $f(x) = x^3$ is $g(x) = \sqrt[3]{x}$ while the additive inverse or negative is $-f(x) = -x^3$ and the multiplicative inverse or reciprocal is $1/f(x) = 1/x^3$.]

A function f that has the same value b for two different numbers a and c in its domain can not have an inverse function g since $f(a) = b = f(c)$, with f and g inverses of each other, implies $a = g(b) = c$. Each of the trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant repeats its values in intervals of 2π and hence does not have an inverse function.

However the trigonometric functions with suitably restricted domains have inverses. For example, as x increases from $-\pi/2$ to $\pi/2$, $\sin(x)$ increases steadily from -1 to 1; hence the sine function with domain restricted to the interval $-\pi/2 \leq x \leq \pi/2$ does not repeat values and so has an inverse function. We use Sine (abbreviated Sin) to denote the sine function with domain the closed interval $[-\pi/2, \pi/2]$ and designate its inverse as Arcsine (Arcsin). The domain of the Arcsine function is [-1, 1] and its range is $[-\pi/2, \pi/2]$.

Similarly, the tangent function on the open interval $-\pi/2 < x < \pi/2$ takes on all real values once and only once and so has an inverse. We designate the tangent function with domain restricted to the open interval $(-\pi/2, \pi/2)$ as Tangent (Tan) and its inverse as Arctangent (Arctan).

As x varies from 0 to π, cos(x) decreases steadily from 1 to -1. Therefore we use Cosine (Cos) to designate the restriction of the cosine function to the domain $[0,\pi]$; its inverse is written as Arccosine (Arccos).

The essential facts about these three inverse trigonometric functions are:

y = Arcsin(x) means that x = sin(y) and $-\pi/2 \le y \le \pi/2$,
y = Arctan(x) means that x = tan(y) and $-\pi/2 < y < \pi/2$,
y = Arccos(x) means that x = cos(y) and $0 \le y \le \pi$.

Frequently Arcsin(x), Arctan(x), and Arccos(x) are written $\sin^{-1}x$, $\tan^{-1}x$, and $\cos^{-1}x$ respectively. One should be prepared for this bad notation in the literature and not allow it to make one confuse an inverse function with a reciprocal. For example, $\sin^{-1}x$ is Arcsin(x) but is **not** $(\sin(x))^{-1}$ = csc(x).

Exercises for Chapter 3 Section 2

1. What might be the motivation for choosing the interval $[0,\pi]$ as the domain of the Cosine function?

2. Give the domain and range of: (a) Arctan(x); (b) Arccos(x).

3. Find Arcsin(½) and four other real numbers x such that sin(x) = ½.

4. Find $\text{Arctan}(\sqrt{3})$ and four other real numbers x such that $\tan(x) = \sqrt{3}$.

5. Find: (a) Arcsin(1); (b) Arctan(1); (c) Arccos(½).

6. Find: (a) $\text{Arcsin}\left(\frac{-\sqrt{2}}{2}\right)$; (b) $\text{Arctan}\left(-\sqrt{3}\right)$; (c) Arccos(-1).

7. (a) Graph both y = Cos(x) and y = Arccos(x) on the same axes.

(b) Are these graphs symmetric to each other with respect to some line?

8. Graph; (a) y = Arcsin(x); (b) y = Arctan(x).

3. Trigonometry on the Calculator.

We note that the calculator has keys labeled SIN, COS, and TAN. These are, as might be expected, the keys for the sine, the cosine, and the tangent functions respectively. If the real number x is on level 1 of the stack, pressing SIN will give $\sin(x)$, TAN will give $\tan(x)$, and COS will give $\cos(x)$. The real number x will be interpreted as degrees or radians according to the angle mode setting. For example if the number 30 is on level 1 of the stack and the calculator is in degree mode, pressing SIN gives the result .5 as expected. If, however, the calculator is in radian mode with 30 on level 1 of the stack, the result of pressing SIN is -.988031624093, which is the sine of 30 radians.

We note that there are no keys for the cosecant, secant, or cotangent functions. Since these functions are the reciprocals of the sine, cosine, and tangent respectively, they can easily be obtained with the keys we have followed by the 1/x key. For information on the trigonometric functions see page 10-8, 12-2, and A-2 of ***UG***.

Calculator Example 3.3.1

Find $\sec(\pi/3)$.

Solution: With the calculator in radian mode key in LS π 3 $\div$ COS 1/x. We see the result 2.00000000001. The answer, of course, should be 2, but we are again seeing an example of the inevitable round off errors caused by using a finite machine to approximate computations with real numbers.

We see that the left shift functions for SIN, COS, and TAN are labeled ASIN, ACOS, and ATAN, respectively. These are, respectively, the Arcsine, Arccosine, and Arctangent functions. These functions are not inverses of each other since, for example, SIN is the sine function, not the Sine function.

Calculator Example 3.3.2

Find $\tan(3\pi/4)$; then take the Arctangent of the result.

Solution: With the calculator in radian mode key in 3 LS π $\times$ 4 $\div$ TAN, and we see the expected answer -1. Now key in LS ATAN and we get the result -.785398163397, which is the decimal approximation for $-\pi/4$. This should not be surprising since $3\pi/4$ is not in the domain of Tan and so $\text{Arctan}(\tan(\theta))$ is not necessarily θ for this angle.

Before starting the the next example, you may want to review the plotting instruction in ***UG***, particularly pages 22-1 to 22-2 and 23-1 to 23-4.

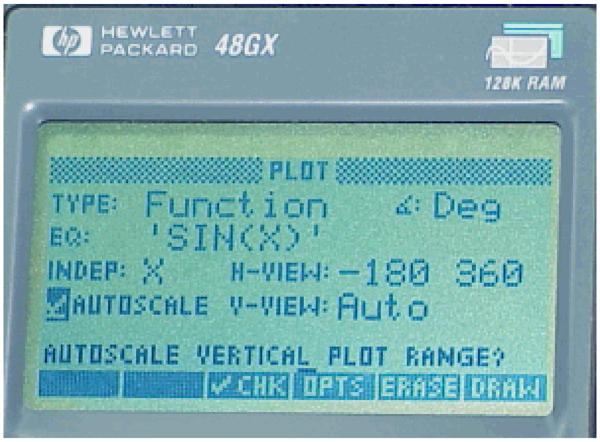
Calculator Example 3.3.3

Graph *y* = sin(*x*) for $-180^\circ \le x \le 360^\circ$ on the calculator.

Solution: The first step is to insure your calculator will react as indicated by these instructions. If you have variables called X, EQ, and/or PPAR in your variable list, purge them. (See page 5-10 of ***UG***.)

Set the calculator to degree mode. Now key in RS PLOT to get into the PLOT dialog box. Press ' SIN m-X ENTER to enter 'SIN(X)' into the EQ: box. Now key in RA 180 +/- ENTER 360 ENTER to set the H-VIEW: boxes. Now press m-CHK to set the V-VIEW: box to AUTO. The dialog box has been set, (see Figure 1a). Now press m-ERASE m-DRAW to draw the graph which we see in Figure 1b.

The tick marks on this graph are still set to the default of 10 pixels. We would like them to be set to 45 on the horizontal axis and .25 on the vertical axis. Press CANCEL m-OPTS to get into the options dialog box. Use DA to get to H-TICK: and change it to 45, change V-TICK: to .25, then press m-CHK to unselect _PIXELS. The options dialog box should now look like Figure 1c. Press m-OK, then m-ERASE and m_DRAW to get the graph as in Figure 1d. Notice that the tick marks now make more sense.



PLOT

TYPE: Function ∡: Deg

EQ: 'SIN(X)'

INDEP: X H-VIEW: -180 360

☑AUTOSCALE V-VIEW: Auto

AUTOSCALE VERTICAL PLOT RANGE?

Figure 1a

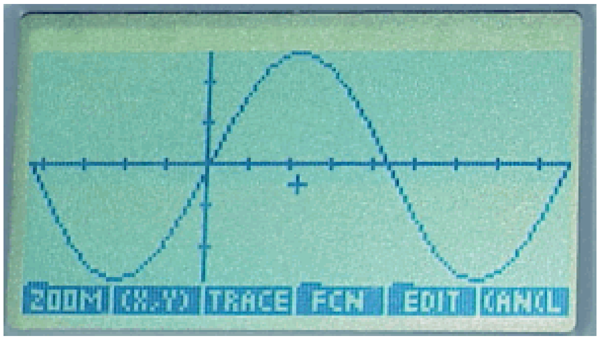
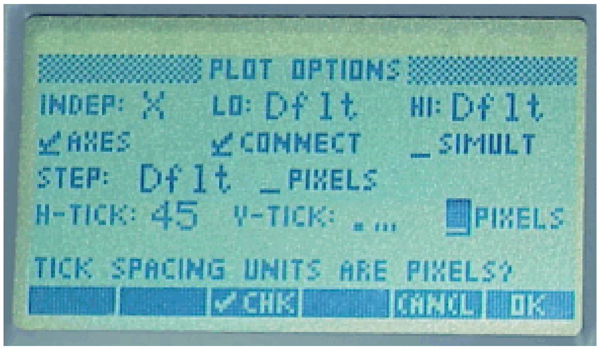


Figure 1b



PLOT OPTIONS

INDEP: X LO: Dflt HI: Dflt

✓AXES ✓CONNECT _SIMULT

STEP: Dflt _PIXELS

H-TICK: 45 V-TICK: [illegible] ■PIXELS

TICK SPACING UNITS ARE PIXELS?

Figure 1c

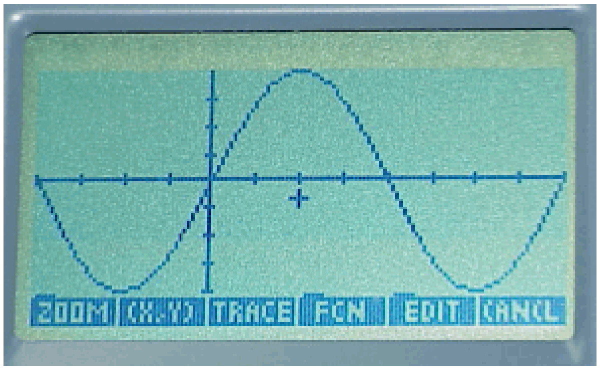


Figure 1d

Calculator Example 3.3.4

Graph y = Cos(x) and y = Arccos(x) on the same axes. See Problem 7, Exercises for Chapter III Section 2.

Solution: Assuming your calculator is as you left it from the previous example, press CANCEL or m-CANCL to return to the PLOT dialog box. Use the arrow keys to get to the angle mode and set it to radians. Go to the EQ: box and put 'COS(X)' in it. Now go to the first field in H-VIEW: and press NXT m-RESET DA m-OK. This resets the plot parameters back to their default state. In this state one unit in both directions is 10 pixels, so geometric properties will not be distorted, but the two views are about twice as big as we need them. We will cut each of the fields in the two view areas in half, and will add .75 to the two V-VIEW fields to move the origin down that amount.

To accomplish this, press m-CALC 2 ÷ m-OK. Now move to the second field in H-VIEW: and repeat. For the two V-VIEW: fields you will insert .75 + after the division. Now press NXT m-OPTS. Set the value of LO: to 0 and move to HI:. We want this value to be π, but we can't put in the symbol, we must enter a decimal approximation. Press NXT m-CALC DROP LS π m-OK. Press m-OK again to get back to the PLOT dialog box then m-ERASE m-DRAW to produce the graph in Figure 2a. Pressing the minus sign will remove the menu so you can see the whole graph. Pressing the minus sign again brings the menu back. Return to the PLOT dialog box and change EQ: to 'ACOS(X)' and in the OPTIONS dialog box change the domain, that is LO: and HI: to -1 and 1 respectively. Now return to the PLOT dialog box and press m-DRAW. **NOTE**: Do not press ERASE in this case since we want this graph to be superimposed on the previous one. See Figure 2b.

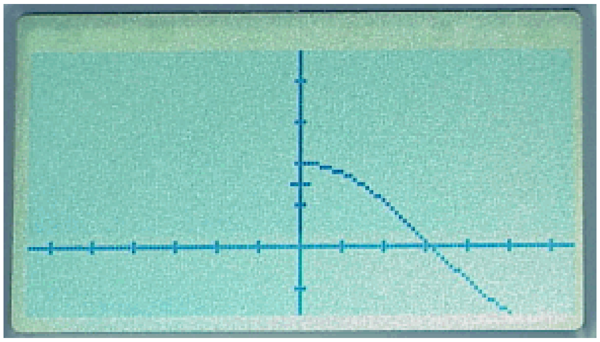


Figure 2a

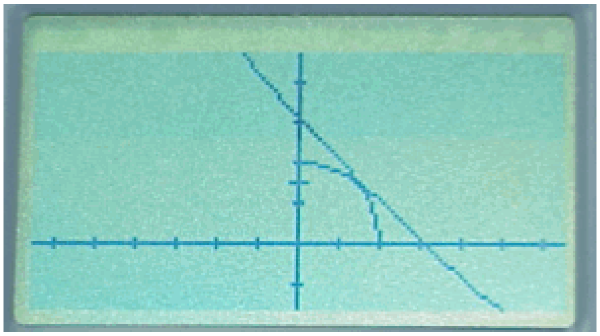


Figure 2b

Finally, to put in the line of symmetry, press m-(X,Y) to show the coordinates at the bottom of the screen and use the arrow keys to move the cursor to X:2 Y:2. Now press m-EDIT NXT m-MARK, then use the arrow keys to move the cursor to the origin.

Now press NXT NXT m-LINE to draw the line, and m-MARK to remove the last mark. You should now have the graph as in Figure 2c.

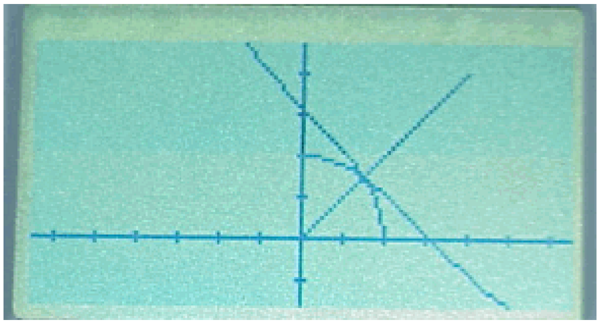


Figure 2c

4. The Laws of Sines and Cosines.

Consider the triangles in Figure 3a and Figure 3b. Since every triangle has at least two acute angles, we can assume we have picked one of them to call α. Since $\sin(\alpha) = \frac{h}{b}$, (see Problem 19, Exercises for Chapter 3 Section 1.) $h = b\sin(\alpha)$. If β is acute as in Figure 3a, then $\sin(\beta) = \frac{h}{a}$. If β is obtuse as in Figure 3b, then $\beta' = 180^{o} - \beta$ and $\sin(\beta) = \sin(180^{o} - \beta) = \sin(\beta') = \frac{h}{a}$. In either case, $h = a\sin(\beta)$. Equating these values of *h*, we have $b\sin(\alpha) = a\sin(\beta)$ or $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b}$.

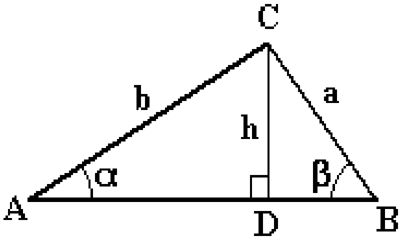


Figure 3a

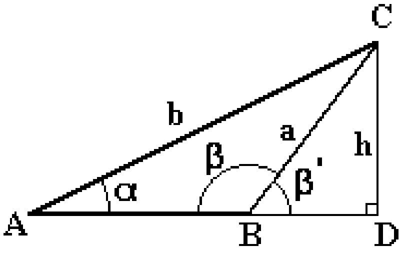


Figure 3b

Similarly, if γ is the angle at *C* and *c* is its opposite side, we can show that $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

We are now prepared to state the ***Law of Sines***: If a triangle has angles α, β, and γ with opposite sides *a*, *b*, and *c* respectively, (See Figure 3c) then

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

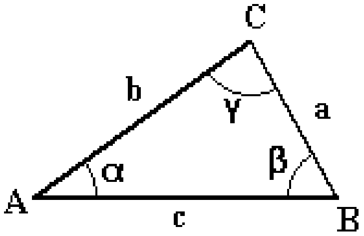


Figure 3c

All of the examples and problems which follow are based on triangles labeled as in Figure 3c.

There are three important congruence theorems from geometry which are usually abbreviated as ASA, SAS, and SSS. ASA, for example, tells us that if two angles and the included side of one triangle are congruent to the respective two angles and included side of another triangle, the two triangles are congruent. Similar statements apply to the other two abbreviations. What these theorems tell us, is that if the right three parts of a triangle are known, the other three parts are fixed and they should be able to be found. Having partial information about a triangle and using it to find the rest of the information about the triangle is referred to as ***solving the triangle***.

Calculator Example 3.4.1

In ΔABC, $\alpha = 37.1^\circ$, $\beta = 58.3^\circ$, and $c = 47.26$. Solve the triangle. Find the angles to one decimal place and the lengths to two decimal places.

Solution: First $\gamma = 180^\circ - \alpha - \beta$, so on the calculator we key 180 ENTER 37.1 - 58.3 - and see the answer $\gamma = 84.6^\circ$. From the law of sines, $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$. We solve this for a and substitute the known quantities to get $a = \frac{47.26\sin(37.1^\circ)}{\sin(84.6^\circ)}$. With the calculator in degree mode and set to Fix 2, we key in 47.26 ENTER 37.1 SIN × 84.6 SIN ÷ and obtain $a = 28.63$. Similarly

$$b = \frac{c\sin(\beta)}{\sin(\gamma)} = \frac{47.26\sin(58.3^\circ)}{\sin(84.6^\circ)} = 40.39.$$

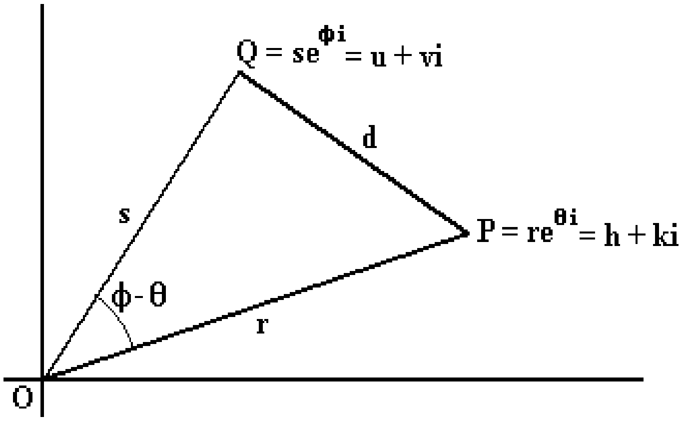


Figure 4

Consider the complex numbers P and Q as shown in Figure 4. By problem 33 of Exercises for Chapter II Sections 4, 5 and 6, we see that d, the distance between P and Q, is

$$d = |Q - P| = \sqrt{(u - h)^2 + (v - k)^2}.$$

But $h = r\cos(\theta)$, $k = r\sin(\theta)$, $u = s\cos(\phi)$, $v = s\sin(\phi)$. Substituting these into the equation for d, squaring both sides, expanding, and collecting like terms, gives us

$$\begin{aligned} d^2 &= \big(s\cos(\phi) - r\cos(\theta)\big)^2 + \big(s\sin(\phi) - r\sin(\theta)\big)^2 \\ &= s^2\big(\cos^2(\phi) + \sin^2(\phi)\big) + r^2\big(\cos^2(\theta) + \sin^2(\theta)\big) - 2sr\big(\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)\big). \end{aligned}$$

Using the Pythagorean Identity [See Problem 7, Exercise for Chapter III Section 1.] and the subtraction formula for the cosine [See Problem 13, Exercise for Chapter III Section 1.] this becomes

$$d^2 = s^2 + r^2 - 2sr\cos(\phi - \theta).$$

If ΔOPQ is now relabeled as in Figure 3c, we get the three following forms of the ***Law of Cosines***

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(\alpha) \\ b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\ c^2 &= a^2 + b^2 - 2ab\cos(\gamma) \end{aligned}$$

depending on whether angle A, B or C, respectively is placed at the origin.

Calculator Example 3.4.2

Given that in ΔABC one has $\gamma = \pi/6$, $a = 22.16$, and $b = 43.26$, solve the triangle. Give angles to three decimal places and lengths to 2 decimal places.

Solution: We observe that the given information is of the form SAS, hence the triangle is determined. We first use the Law of Cosines to find

$$\begin{aligned} c &= \sqrt{a^2 + b^2 - 2ab\cos(\gamma)} \\ &= \sqrt{22.16^2 + 43.26^2 - 2\cdot 22.16\cdot 43.26\cos(\pi/6)}. \end{aligned}$$

On the calculator (in radian mode) this is accomplished with 22.16 LS x^2 43.26 LS x^2 + 2 ENTER 22.16 × 43.26 × LS π 6 ÷ COS × − $\sqrt{x}$, which gives us $c = 26.50$. We now use the Law of Sines to find

$$\sin(\alpha) = \frac{a\sin(\gamma)}{c}$$

or

$$\alpha = \mathrm{Arcsin}\left(\frac{22.16\sin(\pi/6)}{26.50}\right).$$

Assuming the value of *c* is still on the stack from the previous calculation, we proceed with 22.16 ENTER π 6 ÷ SIN × SWAP ÷ LS ASIN, which gives us $\alpha = .431$. (Notice that when we were ready to divide by *c* we took advantage of the fact that it was already on the stack and just used the SWAP command to put it in the right position for the division. Every time one can eliminate the need to key in a number, one has removed an opportunity to make an error.) Finally, we compute the last angle by subtracting the first two from π and get $\beta = 2.187$.

Exercises for Chapter 3 Sections 3 and 4

In all of the following problems give trigonometric functions to 4 decimal places, find all angles in the same units as angles given in the problem with 1 decimal place for degrees and 3 decimal places for radians, and give all lengths with the same number of decimal places as lengths given in the problem.

1. Find the six trigonometric functions for $\alpha = 38.4°$.

2. Find the six trigonometric functions for $\beta = 7\pi/12$.

3. Find the six trigonometric functions for $\gamma = -0.447$ radians.

4. Give each of the following in degrees:

   (a) Arcsin(0.7739); (b) Arcsin(-0.7739); (c) Arccos(0.7739); (d) Arccos(-0.7739);

   (e) Arctan(0.7739); (f) Arctan(-0.7739).

5. Find an angle in radians whose secant is 2.4483.

6. Graph $\cos(\theta)$ for $-\pi/2 \le \theta \le 3\pi/2$ and $\cos(\theta - \pi/6)$ for $-\pi/3 \le \theta \le 5\pi/3$ on the same plot on the calculator.

7. Solve each of the following triangles (See Figure 3c) from the given information if possible.

   (a) $\alpha = 72.3°$, $\beta = 47.6°$, $b = 39.47$;

   (b) $\beta = 5\pi/12$, $a = 2.917$, $c = 3.264$;

   (c) $a = 472.6$, $b = 515.1$, $c = 497.7$, find angles in radians.

8. Reconsider Calculator Example 3.4.2.

   (a) After finding *c*, use the Law of Sines and the ASIN function to find β as α was found in the example.

(b) Why do you get a different answer for β than what was found in the example and what must be done to insure this error does not occur?

9. Solve each of the following triangles:

   (a) $\alpha = 17.9^\circ$, $b = 29.52$, $c = 12.77$;

   (b) $\beta = \pi/12$, $a = 2.932$, $c = 5.761$;

   (c) $\gamma = 0.317$ radians, $a = 139.7$, $b = 62.4$.

10. Show that if you are given two angles and a side which is not the included side, (AAS or SAA), the triangle is still determined. This is usually a corollary to the ASA theorem.

11. Why is AAA not a congruence theorem?

12. Show that SSA does not determine a triangle by finding two distinct triangles which satisfy $a = 2.41$, $b = 4.07$, $\alpha = 32.4^\circ$.