Chapter II

Complex Numbers, A Geometric View

1. Polar Form

The complex number system may be regarded as a numerical representation of the points in a plane (called the ***Argand Plane*** in this context). In the Argand Plane, one selects two points and calls them *O* (***origin***) and *U* (***unity***). The distance between *O* and *U* is chosen as the unit length. Then the location of any other point *P* in the plane is specified by polar coordinates $[r, \theta]$, where *r* is the distance from *O* to *P* and $\theta = \triangle UOP$. The angle θ is positive when measured counterclockwise and negative when measured clockwise. We will generally indicate that *P* has $[r, \theta]$ as polar coordinates by writing $P = re^{\theta i}$; this borrows a notation from Complex Variables courses.

For example, if the distance between *O* and *M* is two units and the angle (measured counterclockwise) from ray *OU* to ray *OM* is $\pi/4$, we write $M = 2e^{(\pi/4)i}$. [See Figure 1a.] Similarly, if the distance between *O* and *N* is ½ and the (clockwise) angle from ray *OU* to ray *ON* is $-2\pi/3$, we write $N = \frac{1}{2}e^{-2\pi i/3}$. [See Figure 1b.] The origin *O* has zero as its *r*-coordinate and any angle may be chosen as its angle; thus $O = 0 \cdot e^{\theta i}$ for all real numbers θ.

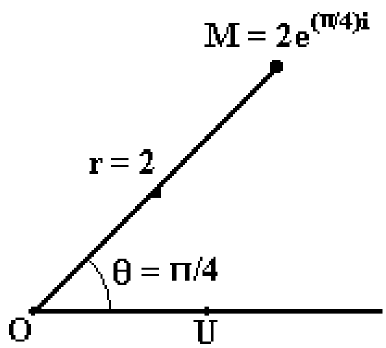

Figure 1a

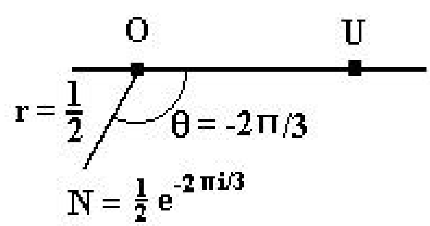

Figure 1b

2. Terminology

The expression $re^{\theta i}$ for a complex number is called its ***polar form***. The nonnegative real number *r* is called the ***absolute value*** or ***modulus*** or ***magnitude*** of *P* and we write $r = |P|$. The angle θ is called an ***argument*** of *P* and is denoted as arg *P*.

A fixed point *P* always has a unique nonnegative real number *r* as its absolute value (i.e., distance to *O*). On the other hand, *P* always has an infinite number of arguments since

$$re^{\theta i} = re^{(\theta \pm 2n\pi)i} \quad \text{for } n = 0,1,2,3,\ldots.$$

For example, some of the other representations for the point $M = 2e^{(\pi/4)i}$ of Figure 1a are $2e^{-7\pi i/4}$, $2e^{9\pi i/4}$, and $2e^{17\pi i/4}$. Also, e^{0i}, $e^{2\pi i}$, $e^{-2\pi i}$, and $e^{-4\pi i}$ are several of the representations of *U*.

The set of points in the Argand Plane is made into the ***Complex Number System*** by defining addition and multiplication as follows:

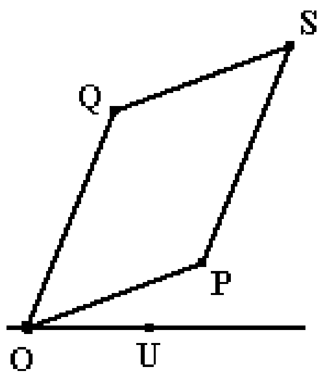

Figure 2

Sum. $P + Q$ is the point *S* such that the directed segment $\overrightarrow{PS}$ has the same magnitude and direction as $\overrightarrow{OQ}$. This means that the equation $S = P + Q$ implies that the quadrilateral *OPSQ* is a parallelogram (See Figure 2) unless it collapses into a line segment.

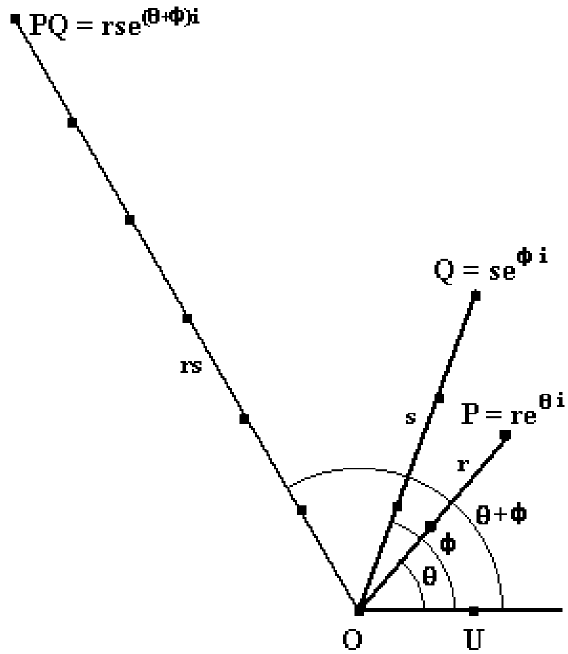

Figure 3

Product. If $P = re^{\theta i}$ and $Q = se^{\phi i}$, then $PQ = rse^{(\theta+\phi)i}$. Thus *PQ* is the point whose absolute value is the product of the absolute values of *P* and *Q* and whose arguments are the sums of an argument of *P* and an argument of *Q*. This is consistent with the fact that the angles θ and ϕ appear in the exponents. See Figure 3.

3. Complex Numbers on the Calculator - Polar Form

Our first job is to get the calculator options set up for the task at hand. Key RS MODES to get into the modes dialog box. There are three settings which we must be sure are properly set: the angle units, the coordinate system, and constant type. Key DA and m-CHOOS to see a drop down list with three choices of angle measure. Use DA or UA as needed to select Degrees from the list then press m-OK. For more information about angle modes see pages 4-3 to 4-4 of ***UG***. Now key DA and m-CHOOS to see three choices for coordinate systems. Again use DA or UA as needed to select Polar from this list then press m-OK. For more information about coordinate systems see pages 4-4 to 4-5 of ***UG***. Finally, we must make sure that the constants in the calculator give numeric rather than symbolic results. Key m-FLAG DA to highlight Flag 2. Now press m-CHK as needed to make sure that Flag 2 is checked then m-OK and m-OK. For more information about the roll of Flag 2 see pages 4-7 and 11-5 of ***UG***.

Now look at the second row of annunciators. The leftmost should be blank indicating that the calculator is in degree mode, and the next one to the right should be $R\angle Z$ indicating that the polar coordinate system is in effect. For more information about the annunciators see pages 1-2 to 1-3 of ***UG***. Notice that the top left key on the calculator, the MTH key, has RAD as its left shifted command and POLAR as its right shifted command. If the setup steps from the previous paragraph were completed correctly, LS RAD will now toggle between radian mode and degree mode. When in radian mode the leftmost annunciator will be RAD. Also, RS POLAR will toggle between polar coordinates and rectangular coordinates. When in rectangular coordinates, the second annunciator is blank.

In the calculator, the complex number $P = re^{\theta i}$ is expressed as $(r, \angle\theta)$. One limitation of the calculator is that it will not accept variables for the magnitude and argument, only numbers. In particular, if $P = 3e^{(\pi/4)i}$, it can't be expressed as $(3, \angle\frac{\pi}{4})$, it must be expressed as $(3, \angle .785398163398)$. As a general rule, this limitation makes working with degrees much easier than with radians. We will, therefore, start with degrees, but we will see some tricks which will make working with radians not quite as bad as it looks. Actually, working in degrees has its own much more subtle problems. Since the calculator actually works with radians internally, there are two conversions in moving data from the keyboard to the display, and these sometimes cause annoying roundoff errors. To avoid this, set your display to Fix 2 for the time being.

Calculator Example 2.3.1

Let $P = (2, \angle 43^o)$ and $Q = (3, \angle 29^o)$, find PQ. With the calculator in degrees and polar coordinates, key in LS ( ) 2 (NOTE: At this point you may key in the comma or a space, SPC, and the calculator will convert it to a comma after you hit ENTER, but neither is necessary.)

RS $\angle$43 ENTER LS ( ) 3 RS $\angle$ 29 ×. You should see $(6.00,\angle 72.00)$ on the display. This clearly satisfies the definition of product from above.

Now try it with $P = (3.4,\angle 87^o)$ and $Q = (6.3,\angle 112^o)$. The result you get is $(21.42,\angle -161.00)$. The magnitude is certainly correct, but we were expecting an argument of 199. What happens is that the calculator always normalizes the argument so that $-180^o < \theta^o \le 180^o$, or in radians, $-\pi < \theta \le \pi$.

Calculator Example 2.3.2

Let $P = (4,\angle 30^o)$ and $Q = (2,\angle 150^o)$ and find $P + Q$. Key in the complex numbers as in the previous example, and press + to find the sum. The result is $(3.46,\angle 60.00)$ to two decimal places. We now use the definition of sum above and geometry to verify that this answer is correct. See Figure 4.

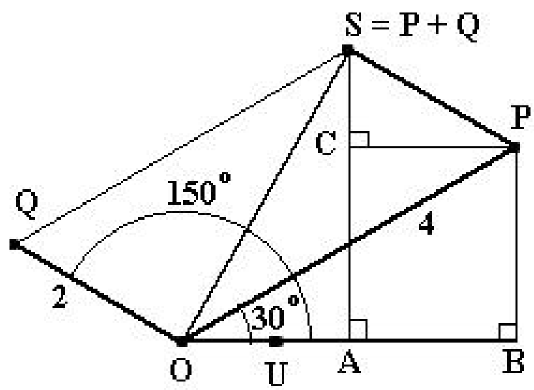

Figure 4

Construct *SA* and *PB* perpendicular to *OU*, and *PC* parallel to *OU*. Now ΔOBP is a 30º, 60º, 90º triangle with hypotenuse 4, so *PB* = 2 = *CA*, and $OB = 2\sqrt{3}$. Since $\angle CPS = 180^o - 150^o = 30^o$, ΔCPS is a 30º, 60º, 90º triangle with hypotenuse 2, so $CP = \sqrt{3} = AB$ and *SC* = 1. *OA* = *OB* - *AB* $= \sqrt{3}$ and *AS* = *AC* + *SC* = 3, so ΔOAS is a 30º, 60º, 90º triangle. Thus $S = (2\sqrt{3},\angle 60^o)$, which agrees with our calculator answer at least to the level of accuracy possible on the calculator.

Calculator Example 2.3.3

An engineer wishes to build a straight railroad from town *A* to town *B* with a tunnel through the mountain from point *C* to point *D*. [See Figure 5] She cannot, of course see *B* from *A*, so she doesn't know what direction to go from *A* to find one end of the tunnel at *C*, nor what direction to go from *B* to find the other end of the tunnel at *D*.

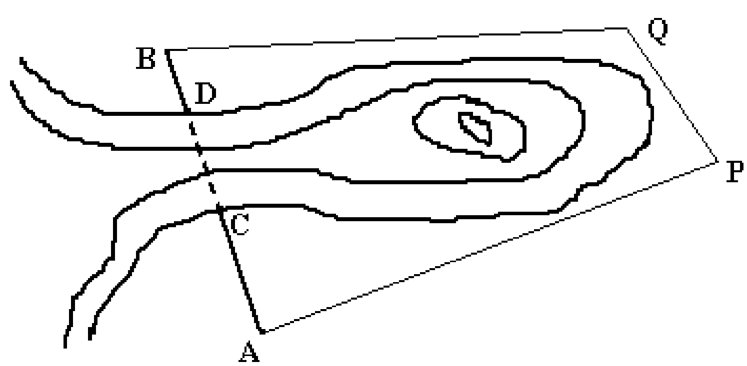

Figure 5

She can, however, see a tall tree at *P* from *A*, from *P* she can see a large rock at *Q*, and from *Q* she can see *B*. How should she proceed?

Solution: She visualizes the map as the Argand Plane with the origin at *A* and *U* one kilometer east of *A*. She finds that from *A* to *P* is 10.13 kilometers in a direction 19.8^{o} measured counterclockwise from east, hence $\overrightarrow{AP}$ can be thought of as the complex number $(10.13,\angle 19.8)$. From *P* to *Q* is 125.3^{o} measured counterclockwise from east and the distance is 3.17 kilometers, so $\overrightarrow{PQ}$ can be thought of as the complex number $(3.17,\angle 125.3)$. Finally, she finds that from *Q* to *B* is 8.68 kilometers and the direction is 177.4^{o} clockwise from east, so $\overrightarrow{QB}$ is considered to be the complex number $(8.68,\angle -177.4)$. She adds these three complex numbers on her calculator and finds the sum to be $(5.71,\angle 99.8)$, so from *A* to *B* is a distance of 5.71 kilometers in the direction 99.8^{o} counterclockwise from east. She now goes in that direction from *A* until she comes to the mountain at a point she labels *C*, then goes in the opposite direction from *B* until she comes to the mountain at the point she labels *D*.

As mentioned above, the calculator's representation of a complex number $P = (r,\angle\theta)$ requires *r* and θ to be expressed in decimal form. If $P = 2\sqrt{3}e^{\frac{\pi}{7}i}$ one can compute the magnitude and argument to twelve significant figures on the calculator, then key them into the parentheses to get $(3.46410161514,\angle .4487989505132)$, but this is a very tedious and error prone procedure. We will consider some tricks for entering such complex numbers more efficiently. For what follows, set the calculator numeric mode to standard to see the full 12 significant digits.

The easiest case is when *r* is "nice" and the argument is of the form $\theta = \frac{a}{b}\pi$, with *b* a divisor of 180. Such an argument is an integer when expressed in degrees, hence easy to enter in that form. For example, if $P = 2e^{\frac{\pi}{4}i}$, the argument is 45^{o}. To get this complex number into the calculator in radian form, first set the calculator to degree mode, enter the number in degree form, then switch the calculator to radian mode; the 45 will change to .785398163398.

We now return to the more general case, $P = 2\sqrt{3}e^{\frac{\pi}{7}i}$. Although there are several ways to handle this, we will discuss only one; the equation writer. (See Chapter 7 of ***UG***.) With the calculator in radian mode, key in LS EQUATION 2 $\sqrt{x}$ 3 RA LS e^x LS π $\div$ 7 RA AS LS I RA EVAL. You should now see $(3.46410161514,\angle .4487989505132)$ on the display. Note that we have used the equation writer to create the algebraic object $2\sqrt{3}e^{\frac{\pi}{7}i}$, then used the evaluation command, EVAL, to convert it to the complex number $(3.46410161514,\angle .4487989505132)$.

Exercises for Chapter 2 Sections 1, 2, and 3

Make all diagrams neat and accurate with the unit of length at least half an inch. It may be helpful to use graph paper and a protractor.

1. Let $C = 2e^{-3\pi i/4}$, $D = 4e^{\pi i/6}$, $E = e^{2\pi i}$, and $F = e^{-2\pi i/3}$. Give the polar form (i.e., $re^{\theta i}$ form) for each of the following products.

   (a) CD; (b) CE; (c) CF; (d) C^2.

2. Do each part of Problem 1 on the calculator (four decimal places). Compare these results with your answers to Problem 1 and explain any apparent discrepancies.

3. Give an alternate representation for $5e^{(\pi/3)i}$ using a negative argument and one using a positive argument different from $\pi/3$.

4. Let $A = 2e^{(5\pi/4)i}$ and $B = 3e^{(-\pi/3)i}$.

   (a) Give the polar form for AB.

   (b) Plot A, B, and AB.

   (c) Construct $S = A + B$ given that $OASB$ is a parallelogram.

5. Let $G = 3e^{0i}$ and $H = 4e^{(\pi/2)i}$. Plot G, H, and $G + H$ and find $|G + H|$, i.e., the distance form O to $G + H$.

6. Let $K = 12e^{(\pi/2)i}$ and $L = 5e^{\pi i}$. Plot K, L, and $K + L$ and find $|K + L|$.

7. Let $M = 4e^{(7\pi/6)i}$ and $N = 2e^{(-\pi/6)i}$.

   (a) Plot M, N, and $M + N$.

   (b) Find $M + N$ using complex numbers on the calculator (four decimal places).

   (c) Use geometric reasoning similar to Calculator Example 2.3.2 to justify the calculator's answer.

8. Let $M = 2\sqrt{3}e^{(\pi/4)i}$ and $N = 2e^{(3\pi/4)i}$.

   (a) Plot M, N, and $M + N$.

(b) Find $M + N$ using complex numbers on the calculator (four decimal places).

(c) Use geometric reasoning to justify the calculator's answer.

9. The points $re^{\theta i}$ for which r is any nonnegative real number and θ is in $\{0,\ 2\pi,\ -2\pi,\ 4\pi,\ -4\pi,\ \ldots\}$ form the ray *OU*. Give similar geometric characterizations for the following sets of points:

(a) The set R of all the points $re^{\theta i}$ with r any nonnegative real number and θ in $\{0,\ \pi,\ -\pi,\ 2\pi,\ -2\pi,\ 3\pi,\ \ldots\}$.

(b) The set I of all the points $se^{\phi i}$ with s any nonnegative real number and ϕ in $\{\pi/2,\ -\pi/2,\ 3\pi/2,\ -3\pi/2,\ 5\pi/2,\ \ldots\}$.

10. Can every point P of the Argand Plane be expressed as $P = A + B$ with A in the set R of Problem 9 (a) and B in the set I of Problem 9 (b)? Explain.

11. Let O be the origin and U be the unity in the Argand Plane and let P, Q, and R be any complex numbers. Verify that the complex number system has each of the following properties:

(a) Commutativity of addition: $P + Q = Q + P$.

(b) Associativity of addition: $(P + Q) + R = P + (Q + R)$. HINT: See Problem 13 of Chapter 1.

(c) Additive identity: $O + P = P + O = P$.

(d) Additive inverse: For each P there exists a complex number N such that $N + P = P + N = O$.

(e) Commutativity of multiplication: $PQ = QP$.

(f) Associativity of multiplication: $(PQ)R = P(QR)$.

(g) Zero multiplication: $OP = PO = O$.

(h) Multiplicative identity: $UP = PU = P$.

(i) Multiplicative inverse: For each $P \neq O$ there exists a complex number M such that $MP = PM = U$.

(j) Distributive law: $P(Q + R) = PQ + PR$. HINT: See Problem 14 of Chapter 1.

12. Let $E = 4e^{0i}$, $F = 4e^{\pi i}$, $G = 7e^{0i}$, and $H = 7e^{\pi i}$. Find the polar form for each of the following:

(a) EG; (b) FH; (c) FG; (d) EH.

13. Do Problem 12 on the calculator and compare with the previous results.

14. Let E, F, G, and H be as in Problem 12. Find the polar form of:

(a) $E + G$; (b) $F + H$; (c) $F + G$; (d) $E + H$.

15. Do Problem 14 on the calculator and compare with the previous results.

16. Let P and Q both be in the set R of Problem 9 (a).

(a) Is the product PQ also in the set R? Explain.

(b) Is the sum $P + Q$ also in R? Explain.

(c) Is $Qe^{(\pi/2)i}$ in the set I of Problem 9 (b)? Explain.

17. Let $P = 6e^{(4\pi/9)i}$. Find the polar form of a point N such that $P + N = O$. Show P,O, and N in a diagram.

18. Let $Q = 4e^{(5\pi/3)i}$ and $\overline{Q} = 4e^{(-5\pi/3)i}$.

(a) Show Q, $\overline{Q}$, $Q\overline{Q}$, and $Q + \overline{Q}$ in a diagram.

(b) Find $Q\overline{Q}$ and $Q + \overline{Q}$ on the calculator and compare with your diagram from (a).

4. Negative of a Point, Subtraction, Conjugate

As in other number systems, if $P + N = O$, one writes $N = -P$ and calls N the ***negative*** of P. It follows from the definition of addition of points in the Argand Plane that N is the point such that the directed line segment $\overrightarrow{ON}$ has the same magnitude and direction as $\overrightarrow{PO}$; i.e., N is the point such that O is the midpoint of segment PN. If $P = re^{\theta i}$, then clearly $N = re^{(\theta+\pi)i} = re^{(\theta-\pi)i}$. See Figure 6.

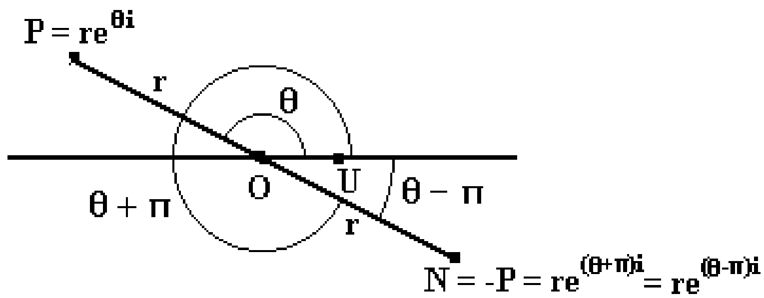

Figure 6

The ***difference*** E - F is the point G such that $E = F + G$. One can also obtain the difference $G = E - F$ using the formula $G = E + (-F)$. See Figure 7.

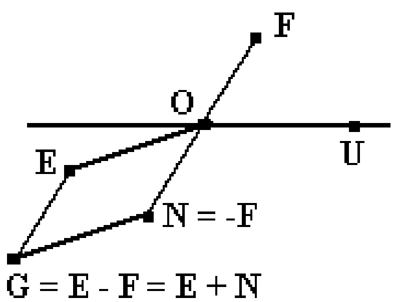

Figure 7

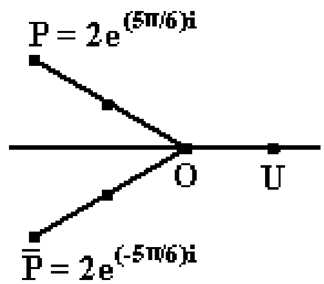

Figure 8

In the Argand Plane, the ***conjugate*** of a point $P = re^{\theta i}$ is the point $\overline{P} = re^{-\theta i}$, with the same absolute value as P but with the argument the negative of that of P. Note that a point P and its conjugate $\overline{P}$ are symmetrically situated with respect to the straight line determined by O and U. See Figure 8.

These concepts are all easily handled on the calculator. The +/- key converts the complex number in level 1 of the stack to its negative. Enter your favorite complex number and try it! Note that pressing +/- twice returns the original number as it should. The - key will subtract the complex number in level 1 from the complex number in level 2. Try it with your choice of complex numbers A and B. After computing $S = A - B$, verify that $S + B$ returns your original A. There is also a conjugate function, but it is in one of the menus. Key in MTH NXT m-CMPL NXT and you will find CONJ as the last item in the menu. Key in a complex number and try it. Again, pressing it twice gives back the original complex number as expected.

There are two other functions on this menu page; NEG and SIGN. NEG works the same as the +/- key and SIGN converts the complex number on level 1 into a complex number with the same argument but with absolute value 1. Press NXT and you see six more functions related to complex numbers. The first four will be discussed after the next section, but ABS returns the absolute value of the complex number on level 1, and ARG returns its argument.

5. The Real and Imaginary Axes; Rectangular Form

Each point on the straight line through the origin *O* and the unity point *U* is expressible as $re^{\theta i}$ with $\theta = 0$ or π. The set of all these points is closed under addition and multiplication. (See Problem 16 above.) In fact, these points behave just like the real numbers under addition, subtraction, multiplication, and division. For this reason, the line through *O* and *U* is called the ***real axis*** and we identify each real number with a point on the real axis in the following manner.

The real number zero is identified with the origin *O*. A positive (real) number *r* is identified with the point re^{0i}. The material in the previous section makes it natural for its negative $-r$ to represent the point $re^{\pi i}$. In particular, we have $U = e^{0i} = 1$.

The line perpendicular to the real axis at the origin is called the ***imaginary axis***. The imaginary unit point $e^{(\pi/2)i}$ is designated as i. Then the points $re^{(\pi/2)i}$ may be written as ri and points $re^{(3\pi/2)i}$ as $-r$i. An important fact is that $i^2 = e^{\pi i/2}e^{\pi i/2} = e^{\pi i} = -1$; that is, $i^2 = -1$.

Now every point on the real axis is represented by a real number *a* and every point on the imaginary axis by a ***pure imaginary number***, i.e., a number *b*i with *b* real. Note that *a* and *b* may be positive, zero, or negative. See Figure 9.

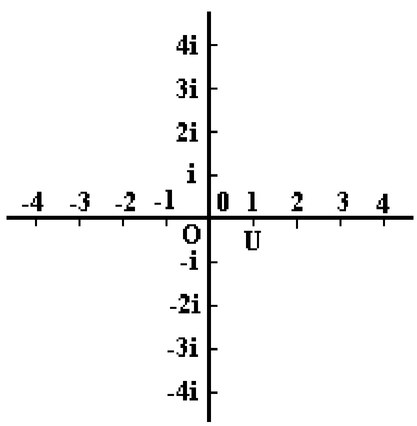

Figure 9

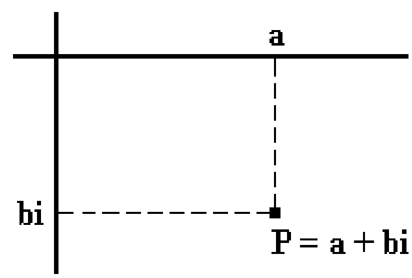

Figure 10

Let *P* be any point in the Argand Plane. Then the foot of the perpendicular from *P* to the real axis has a representation as some real number *a*. Similarly, the foot of the perpendicular from *P* to the imaginary axis is a pure imaginary number *b*i. The rule for adding points shows that $P = a + b$i. [The parallelogram with vertices 0, *a*, *P*, *b*i turns out to be a rectangle in this case. See Figure 10.]

The representation $a + b$i, with a and b real, is called the ***rectangular form*** of a complex number. The real number a is called the ***real part*** and the real number b is called the ***imaginary part*** of the complex number.

It can be shown that in rectangular form the complex numbers have the following rules:

ADDITION $\quad (a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}.$

SUBTRACTION $\quad (a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}.$

MULTIPLICATION $\quad (a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}.$

EXAMPLE 1. Convert the polar form $P = 5\sqrt{2}\,e^{(3\pi/4)i}$ to rectangular form $a + b$i.

Solution: Let H and V be the feet of the perpendiculars from P to the real axis and the imaginary axis, respectively. We see that ΔPHO is a 45°, 45°, 90° triangle. Hence the lengths of its sides are in the ratio $1{:}1{:}\sqrt{2}$. Since the hypotenuse has length $5\sqrt{2}$, the two equal sides must have length 5. Then $H = 5e^{\pi i} = -5$ and $V = 5e^{(\pi/2)i} = 5i$. Hence $P = H + V = -5 + 5i$. See Figure 11.

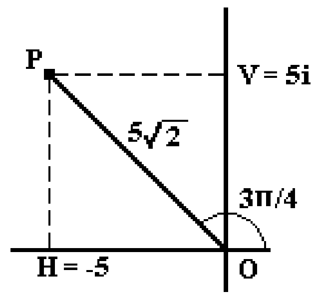

Figure 11

EXAMPLE 2. Convert the rectangular form $Q = -7 - 7\sqrt{3}i$ to polar form.

Solution: Let C = -7 and $D = -7\sqrt{3}i$. [See Figure 12.] Clearly ΔQDO is a right triangle. Since the ratio of the length of side DO to the length of the side QD is $\sqrt{3}$, it is a 30°, 60°, 90° triangle and the hypotenuse OQ has twice the length of the shortest side QD, that is, the hypotenuse has length 14. Also, the counterclockwise angle from ray OU to ray OQ is 240° i.e., $4\pi/3$. Hence $Q = 14e^{(4\pi/3)i}$.

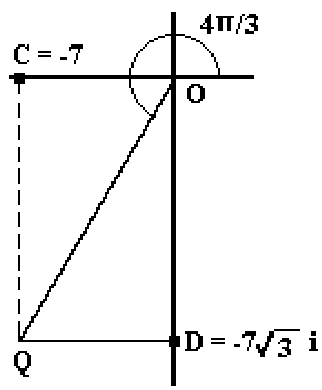

Figure 12

EXAMPLE 3. Find the rectangular form of
$P = 2\sqrt{2}e^{(5\pi/12)i}$.

Solution: First we note that $5\pi/12$ radians equals 75°, or 30° plus 45°. Using a 30°, 60°, 90° triangle and a 45°, 45°, 90° triangle [see Figure 13], one finds that

$$\sqrt{2}e^{(\pi/4)i} = 1 + i,$$

$$2e^{(\pi/6)i} = \sqrt{3} + i.$$

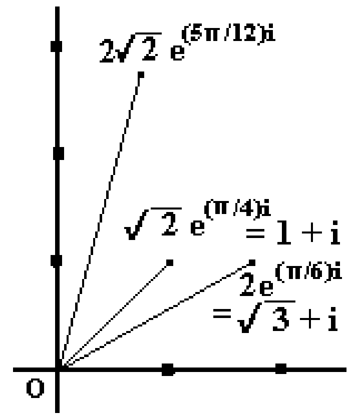

Figure 13

Multiplying these two complex numbers, one has

$$\sqrt{2}e^{(\pi/4)i}2e^{(\pi/6)i} = (1 + i)(\sqrt{3} + i)$$

$$2\sqrt{2}e^{(5\pi/12)i} = (\sqrt{3} - 1) + (\sqrt{3} + 1)i.$$

6. Rectangular Form on the Calculator

The complex number $a + b$i has rectangular form (a,b) on the calculator. For example, to enter 3 + 5i into the calculator make sure it is in rectangular coordinate mode then key in LS ( ) 3 SPC (one could key in the comma here instead of SPC, but that would take two key strokes and the calculator will turn the space into a comma anyway) 5 ENTER. At this point we can also discuss the first four functions in the complex number menu which were mentioned earlier. Key MTH NXT m-CMPL to get back into the complex number menu. Assuming there is a complex number on level 1 of the stack (with the calculator in either rectangular or polar mode) m-RE returns the real part of that complex number and m-IM returns the imaginary part of the complex number on level 1. The function m-C->R takes the complex number from level 1 and puts the real part on level 2 and the imaginary part on level 1 while m-R->C reverses that process. It takes a real number from level 2 as the real part and a real number from level 1 as the imaginary part and forms a complex number which it puts on level 1.

Calculator Example 2.6.1

We will use the calculator to do Examples 1 and 2 above. For Example 1, we were to find the rectangular form of $P = 5\sqrt{2}e^{(3\pi/4)i}$. With the calculator set to rectangular coordinates and standard display mode, use the equation writer to key in the polar form of *P*: LS EQUATION 5 $\sqrt{x}$ 2 RA LS e^x 3 LS π ÷ 4 RA AS LS I RA EVAL. The result will be (-4.99999999998,5). The real part should, of course, be -5, but we are seeing the unavoidable roundoff errors which occur when we use a finite device to approximate real numbers.

For Example 2 we were to find the polar form for $Q = -7 - 7\sqrt{3}i$. We will place the real part on level 2 of the stack and the imaginary part on level 1, then use the R->C command to create the complex number. Set the calculator to rectangular coordinates, standard display mode, degree angle mode, and bring up the complex number menu. Now key 7 +/- ENTER ENTER 3 $\sqrt{x}$ ×m-R->C. We should now see (-7,-12.124355653) on the display. Now RS POLAR converts it to polar form and we see $(14,\measuredangle -120)$, and after LS RAD we convert it to radians and see $(14,\measuredangle -2.09439510239)$. There is one more thing we can try which sometimes gives us an answer in a form we would like to see it. Key m-ARG LS SYMBOLIC NXT $\text{m}\!-\!\to\!Q\pi$ and we see $'-(2/3*\pi)'$ on the display. The $\text{m}\!-\!\to\!Q\pi$ command works to convert a decimal expression into a rational number times π if it's not to far from one of the "nice" angles. If, however, you compute $\frac{3\pi}{7}$ as a decimal then try $\text{m}\!-\!\to\!Q\pi$ to get it back, you get a very strange result.

Calculator Example 2.6.2

Find |(3 + 7i)(2 - 5i)|. We will first get back into the complex number menu with MTH NXT m-CMPL. We now key in the problem: LS ( ) 3 SPC 7 ENTER LS ( ) 2 SPC 5 +/- × m-ABS. We see the answer 41.0121933088. Notice that it doesn't matter if the calculator is in rectangular or polar coordinates when we key in the problem.

Exercises for Chapter 2 Sections 4, 5, and 6

1. Convert the following polar forms to rectangular form $a + bi$. Do each part both by hand (exact answer) and on the calculator (three decimal approximation) and compare results. For the calculator part, enter the number in polar form then switch to rectangular form.

   (a) $8e^{(\pi/6)i}$; (b) $8e^{(5\pi/6)i}$; (c) $\sqrt{2}e^{(5\pi/4)i}$; (d) $9e^{(3\pi/2)i}$; (e) $4e^{(-\pi/4)i}$.

2. Convert the following rectangular forms to polar form $re^{\theta i}$. Do each part both by hand (exact answer) and on the calculator (three decimal approximation) and compare results. For the calculator part, enter the numbers in rectangular form then switch to polar form. Give angles in radians.

   (a) 1 + i; (b) 7i; (c) $-5\sqrt{3} + 5i$; (d) -3; (e) -4 - 4i.

3. Find the rectangular form $a + bi$ of $e^{\theta i}$ for

$$\theta = 0,\ \pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6,\ \text{and}\ \pi.$$

4. Do as in Problem 3 for $\theta = 7\pi/6,\ 5\pi/4,\ 4\pi/3,\ 3\pi/2,\ 5\pi/3,\ 7\pi/4,\ 11\pi/6,\ \text{and}\ 2\pi$.

5. Find the rectangular form for the conjugate $\overline{P}$ of $P = -7 + 8i$. Do by hand and on the calculator.

6. Find the rectangular form for the conjugate $\overline{P} = re^{-\theta i}$ of $P = re^{\theta i} = a + bi$, where a and b are real numbers.

7. Let $S = 13e^{\phi i} = -12 + 5i$. Find both the polar and the rectangular form of the point symmetric to S with respect to:

(a) the real axis; (b) the imaginary axis (c) the origin

(d) the straight line through O and $1 + i$.

8. Do the same as in Problem 7 for $T = re^{\theta i} = h + ki$ instead of S.

9. Let $A = 5e^{\alpha i} = 3 + 4i$ and $B = 6\sqrt{2}e^{(3\pi/4)i} = -6 + 6i$. Find in both polar and rectangular form and in each case give both the exact answer and a four decimal approximation from the calculator:

(a) $-A$; (b) $-B$ (c) $\overline{A}$; (d) $\overline{B}$; (e) AB; (f) $A\overline{B}$.

10. For A and B of Problem 9, give the rectangular form of $A - B$ and plot O, A, $A - B$, and $-B$. What kind of quadrilateral are these four points the vertices of?

11. Let $A = 5e^{\alpha i} = 3 + 4i$ and $D = 13e^{\beta i} = 12 - 5i$. Find the rectangular form of :

(a) $65e^{(\alpha+\beta)i}$; (b) $65e^{(\alpha-\beta)i}$.

12. Let $B = 6\sqrt{2}e^{(3\pi/4)i} = -6 + 6i$ and $C = 2e^{(\pi/6)i} = \sqrt{3} + i$. Find $12\sqrt{2}e^{(7\pi/12)i}$ in rectangular form both as an exact value and as a 4 decimal calculator approximation.

13. Given that $re^{\theta i} = 7 + 5i$, use the Pythagorean Theorem to find r.

14. Given that $8e^{\phi i} = 5 + bi$ and $b < 0$, find b.

15. Explain why $|a + bi| = \sqrt{a^2 + b^2}$ for all real a and b and find $|11 - 8i|$ both as an exact value and as a four decimal calculator approximation.

16. Let $P = re^{\theta i} = a + bi$ and $Q = se^{\phi i} = c + di$. Show that :

(a) $\overline{P + Q} = \overline{P} + \overline{Q}$; (b) $\overline{PQ} = \overline{P}\cdot\overline{Q}$.

17. Let $A = 5e^{\alpha i} = 3 - 4i$. Find in both polar and rectangular form:

(a) iA; (b) $-A$; (c) $-iA$; (d) $\overline{A}$; (e) $i\overline{A}$; (f) A^2; (g) A^3.

18. Let $B = 29e^{\beta i} = -21 + 20i$. Find the polar form (in terms of β) for :

(a) -20 - 21i; (b) 21 - 20i; (c) 20 + 21i; (d) -21 - 20i; (e) 20 - 21i.

19. Let $P = 5 - 5\sqrt{3}i$. Find P, $\overline{P}$, $-P$, and $-\overline{P}$ in both polar and rectangular form.

20. Let $Q = 2\sqrt{2}e^{(3\pi/4)i}$. Find Q, $\overline{Q}$, $-Q$, and $-\overline{Q}$ in both polar and rectangular form.

21. Given that $P = 13e^{\theta i} = 12 - 5i$ and $Q = 5e^{\phi i} = 4 + 3i$, find in rectangular form:

(a) $\overline{P} = 13e^{-\theta i}$; (b) $Q\overline{P} = 65e^{(\phi-\theta)i}$; (c) $-\overline{P} = 13e^{(\pi-\theta)i}$;

(d) $Q^2 = 25e^{2\phi i}$; (e) $PQ = 65e^{(\theta+\phi)i}$.

22. Let $A = 25e^{\alpha i} = 7 + 24i$ and $B = \sqrt{13}e^{\beta i} = 2 - 3i$. For each of the following angles θ, find r, a, and b such that $re^{\theta i} = a + bi$:

(a) $\theta = \alpha - \beta$; (b) $\theta = 2\beta$; (c) $\theta = \alpha - \pi$; (d) $\theta = \alpha + (3\pi/2)$.

23. Express $(\sqrt{3} + i)^{10}(2 - 2\sqrt{3}i)^9$ in $a + bi$ form by first converting $\sqrt{3} + i$ and $2 - 2\sqrt{3}i$ to polar form, raising to powers and multiplying in polar form, and finally converting back.

24. Do as in Problem 23 for $(5 + 5i)^{11}(-7i)^8$.

25. Let $A = 1024e^{(\pi/4)i}$ and $C = e^{(2\pi/5)i}$.

(a) Find in polar form a complex number B such that $B^5 = A$.

(b) Find in polar form and plot B, CB, C^2B, C^3B, C^4B, and C^5B.

26. Find in polar form and plot 5 fifth roots of $243e^{(-13\pi/18)i}$. What kind of geometrical figure has these 5 points as vertices?

27. Let $P = 13e^{\theta i} = 5 + 12i$, $Q = 13$, and $S = P + Q$.

(a) Explain why O, Q, S, P are the vertices of a rhombus (i.e., a parallelogram with all four sides equal).

(b) Explain why $\angle QOS = \frac{1}{2}\angle QOP$.

(c) Find the absolute value s of S and then find $R = \left(\frac{\sqrt{13}}{s}\right)S$ in $a + bi$ form.

(d) Let R be as in (c). Explain why R and $-R$ are the 2 square roots of P.

28. Let $P = re^{\theta i} = a + bi$, $Q = r$, $S = P + Q$, and $s = |S|$. Explain why each of the following is true.

(a) $O, Q, S,$ and P are the vertices of a rhombus.

(b) $\angle QOS = \frac{1}{2}\angle QOP$.

(c) $r^2 = a^2 + b^2$, $s^2 = 2r^2 + 2ra$, and $s = \sqrt{r}\sqrt{2r + 2a}$.

(d) The square roots of P are $\pm\left(\frac{\sqrt{r}}{s}\right)S = \pm\frac{S}{\sqrt{2r + 2a}} = \pm\frac{P + r}{\sqrt{2r + 2a}}$.

29. Let $P = re^{\theta i} = a + bi$ with $b > 0$. Use Problem 28 to show that the 2 square roots of P are $\pm\left(\sqrt{\frac{r + a}{2}} + i\sqrt{\frac{r - a}{2}}\right)$.

30. Let $P = re^{\theta i} = a + bi$ with $b < 0$. Use Problem 28 to show that the 2 square roots of P are $\pm\left(\sqrt{\frac{r + a}{2}} - i\sqrt{\frac{r - a}{2}}\right)$.

31. Let $V = e^{\theta i} = c + s\text{i}$. Show that the square roots $e^{\theta i/2}$ and $e^{(\theta+2\pi)i/2}$ of V are

$$\pm\sqrt{\frac{1+c}{2}} \pm \text{i}\sqrt{\frac{1-c}{2}}$$

where like signs are used if $s > 0$ and unlike signs if $s < 0$.

32. Use Problem 31 to find $e^{(\pi/8)i}$ in $a + b$i form.

33. Let $D = B - A$. See figure 14.

(a) Explain why $\vec{OD}$ and $\vec{AB}$ have the same magnitude and hence $|B - A|$ equals the distance between A and B.

(b) If $A = u + v$i and $B = x + y$i, show that

$$|B - A| = \sqrt{(x-u)^2 + (y-v)^2}.$$

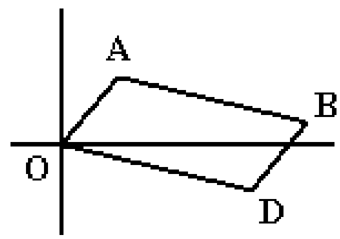

Figure 14

34. Let $A = 5 + 2$i, $B = 9 + 5$i, and $C = 5 + 5$i.

(a) Explain why $\angle ACB$ is a right angle.

(b) Find sides AC and CB of ΔABC and then use the Pythagorean Theorem to find side AB (i.e., the distance between A and B.)

(c) Find $|B - A|$ and $|B| - |A|$ and tell which equals the distance between A and B.

35. Sketch and identify the locus of all points A in the Argand Plane such that $|A - \text{i}| = 3$. (That is, give the graph in the Argand Plane of the equation $|A - \text{i}| = 3$.)

36. Do as in Problem 35 for each of the following equations.

(a) $|A - 4| + |A - 3\text{i}| = 5$.

(b) $|A - 2\text{i}| - |A + 2\text{i}| = 1$.

37. Let $P = re^{\theta i} = a + bi$ and $Q = se^{\theta i} = c + di$ be two complex numbers with the same argument θ. Also let $r > 0$ and $s > 0$. Use similar right triangles to explain why the following are true:

$$\frac{a}{r} = \frac{c}{s}, \qquad \frac{b}{r} = \frac{d}{s}, \qquad \frac{b}{a} = \frac{d}{c}.$$

38. Let *P* be as in Problem 27. With the calculator in rectangular coordinates key in *P* and press the $\sqrt{x}$ key. Compare to the results from Problem 27.

39. Let *A* and *C* be as in Problem 25.

(a) Use the calculator in degree mode to find $B = A^{1/5}$ by making use of the y^x key.

(b) Compare your answer to part (a) with the value of *B* from Problem 25 (a).

(c) Use the calculator to repeat the computations (but not the plots) from Problem 25 (b).

40. Use the calculator to repeat the computations, but not the plots, from Problem 26.

41. Let *A*, *B*, and *C* be as in Problem 34. Use the m-ABS function in the complex number menu to find the distances between points *A* and *B*, between *A* and *C*, and between *B* and *C*.