

## Supplementary Problems

These problems are intended for more advanced students. Some of them require concepts from Calculus and Linear Algebra.

1. Use the identity  $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$  to solve each of the following equations:

(a)  $4x^3 - 3x = 1/2$ ;    (b)  $8x^3 - 6x - \sqrt{3} = 0$ ;    (c)  $x^3 - 3x + 1 = 0$ .

2. Convert each of the following functions into the form  $r\sin(\theta - \alpha)$ , with appropriate constants  $r$  and  $\alpha$ , and thus find the maximum and minimum values of  $f(\theta)$  without using differentiation. Check using calculus.

(a)  $f(\theta) = 2\sin(\theta) - 2\sqrt{3}\cos(\theta)$ ;

(b)  $f(\theta) = 20\cos(\theta) + 21\sin(\theta)$ .

3. Prove the following identities and generalize:

(a)  $\frac{\sin(\theta)}{\sin(\frac{\theta}{8})} = 8\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)$ ;

(b)  $\frac{\sin(\theta)}{\sin(\frac{\theta}{9})} = \left[4\cos^2\left(\frac{\theta}{3}\right) - 1\right] \cdot \left[4\cos^2\left(\frac{\theta}{9}\right) - 1\right]$ .

4. Let  $P = e^{\theta i} = \cos(\theta) + i\sin(\theta)$ . Show the following:

(a)  $P^n = \cos(n\theta) + i\sin(n\theta)$ ;

(b)  $1 + P + P^2 + \dots + P^{n-1} = \frac{1 - P^n}{1 - P}$ ;

(c)  $1 + \cos(\theta) + \cos(2\theta) + \dots + \cos[(n-1)\theta] =$

$$\frac{[1 - \cos(\theta)][1 - \cos(n\theta)] + \sin(\theta)\sin(n\theta)}{2 - 2\cos(\theta)};$$

$$(d) \frac{\sin(\theta) + \sin(2\theta) + \dots + \sin(n\theta)}{\cos(\theta) + \cos(2\theta) + \dots + \cos(n\theta)} = \tan\left[\frac{(n+1)\theta}{2}\right].$$

5. Use the Euler Formula  $e^{\theta i} = \cos(\theta) + i\sin(\theta)$  to express
- $\cos(\theta)$  in terms of  $e^{\theta i}$  and  $e^{-\theta i}$ ;
  - $\sin(\theta)$  in terms of  $e^{\theta i}$  and  $e^{-\theta i}$ .
6. This problem assumes familiarity with Calculus.
- Find the Taylor series about 0 of  $\cos(x)$  and  $\sin(x)$ . Note: It is known that both of these series still converge if the real number  $x$  is replaced by any complex number  $z$ .
  - Use the results of part (a) and the Euler Formula to find a series for  $e^{xi}$ .
  - Replace  $x$  with  $-i$  in the result of part (b) and simplify the resulting series. How does the "e" introduced in this text compare to the "e" (that is, the base of the natural logarithm) introduced in Calculus?
7. Prove that  $2\sin\left(\frac{\alpha - \beta}{2} + \frac{\pi}{4}\right)\cos\left(\frac{\alpha + \beta}{2} + \frac{\pi}{4}\right) = \sin(\beta) + \cos(\alpha)$ .
8. Show that  $\sin(x + y + z) = \sin(x)\cos(y)\cos(z) + \cos(x)\sin(y)\cos(z) + \cos(x)\cos(y)\sin(z) - \sin(x)\sin(y)\sin(z)$ .
9. Show that  $\cos(x + y + z) = \cos(x)\cos(y)\cos(z) - \sin(x)\sin(y)\cos(z) - \sin(x)\cos(y)\sin(z) - \cos(x)\sin(y)\sin(z)$ .
10. (a) With the calculator in radian mode put 'SIN(X)' on level 2 of the stack and, 'X' on level 1, then press RS  $\partial$ .
- Leave the result of (a) on the stack but change the mode to degrees and repeat part (a).
  - Why do you get different results to parts (a) and (b)?

Problems 11 and 12 assume familiarity with Linear Algebra.

11. Let the linear operator  $T(\mathbf{v})$  be defined by  $T(\mathbf{v}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} \mathbf{v}$ , let  $\Delta OAB$  be as in

Problem 42 of Exercises for Chapter 2 Sections 3, 4, and 5, let  $\mathbf{v} = \vec{OA}$ , and let  $\mathbf{w} = \vec{OB}$ . Verify that  $T(\mathbf{v})$  and  $T(\mathbf{w})$  have the same effect on  $\Delta OAB$  as multiplying  $A$  and  $B$  by  $e^{(2/3)\pi i}$  had on that triangle. In this sense, the linear operator  $T$  and multiplying by  $e^{(2/3)\pi i}$  are equivalent operations.

12. Show that the linear operator  $T(\mathbf{v}) = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{v}$  and multiplying by  $re^{\theta i}$ ,  $r > 0$ , are equivalent processes in the sense given in Problem 11 above.

13. Sketch and identify the locus of all points  $A$  in the Argand Plane that satisfy the following equations:

(a)  $|A - 3i| + |A + 3i| = 10$ .

(b)  $|A - 3i| - |A + 3i| = 1$ .

## Answers and Hints for Selected Odd Numbered Problems

### Exercises for Chapter I

1. acute;  $\sqrt{37}^\circ$ ; right;  $90^\circ$ ; obtuse;  $156^\circ$ : The rest are not possible in a triangle.

3. (i) (a)  $3\sqrt{2}$ ; (b)  $\frac{11}{2}\sqrt{3}$ ; (c) 8 (d)  $\sqrt{33}$ .

(ii) (a) 4.24 (b) 9.53 (c) 8.00 (d) 5.74

5.  $x = 9/2$   $y = 6$ .

7. Yes

9. (a)  $u = \frac{a}{r^2}$ ;  $v = \frac{b}{r^2}$ .

(b)  $r = 2.823$ ;  $u = 0.326$ ;  $v = 0.138$

13. Hint: Make repeated use of the theorem in Section 5.

15. Hint: Use Problem 14.

17. (a) 13 (b) 15 (c)  $\frac{\pi}{6}$ ; (d)  $\frac{\pi}{4}$ .

### Exercises for Chapter II Sections 1 and 2

1. (a)  $8e^{-7\pi i/12}$ ; (b)  $2e^{5\pi i/4}$ ;  $2e^{-17\pi i/12}$ ; (d)  $4e^{-3\pi i/2}$ .

3. (a)  $6e^{11\pi i/12}$ .

5.  $|G + H| = 5$ .

9 (a) The line  $OU$ . (b) The line through  $O$  perpendicular to  $OU$ .

11.  $6e^{13\pi i/9}$ .

15. (a)  $16e^{\pi i}$ ; (b)  $28e^{2\pi i}$ ; (c)  $28e^{\pi i}$ ; (d)  $28e^{\pi i}$ .

17. (a) Yes; (b) Yes; (c) Yes.

19 (b)  $4e^{\pi i/20}$ .

**Exercises for Chapter II Sections 3, 4, and 5**

1. (a)  $4\sqrt{3} + 4i$ ; (b)  $-4\sqrt{3} + 4i$ ; (c)  $-1 - i$ ;

(d)  $-9i$ ; (e)  $2\sqrt{2} - 2\sqrt{2}i$ .

5.  $-7 - 8i$ .

7. (a)  $13e^{-\Phi i} = -12 - 5i$ ; (b)  $13e^{(\pi-\Phi)i} = 12 + 5i$ ;

(c)  $13e^{(\Phi+\pi)i} = 12 - 5i$ ; (d)  $13e^{\left(\frac{\pi}{2}-\Phi\right)i} = 5 - 12i$ .

9. (a)  $5e^{(\alpha+\pi)i} = -3 - 4i$ ; (b)  $6\sqrt{2}e^{(7\pi/4)i} = 6 - 6i$ ;

(c)  $5e^{-\alpha i} = 3 - 4i$ ; (d)  $6\sqrt{2}e^{-(3\pi/4)i} = -6 - 6i$ ;

(e)  $30\sqrt{2}e^{(\alpha+3\pi/4)i} = -42 - 6i$ ; (f)  $30\sqrt{2}e^{(\alpha-3\pi/4)i} = 6 - 42i$ .

11. (a)  $56 + 33i$ ; (b)  $16 + 63i$ .

13.  $\sqrt{74}$ .

15.  $\sqrt{185}$ .

19. (a)  $5e^{(\alpha+\pi/2)i} = 4 + 3i$ ; (b)  $5e^{(\alpha+\pi)i} = -3 + 4i$ ;

(c)  $5e^{(\alpha-\pi/2)i} = -4 - 3i$ ; (d)  $5e^{-\alpha i} = 3 + 4i$ ;

(e)  $5e^{\left(\frac{\pi}{2}-\alpha\right)i} = -4 + 3i$ ; (f)  $25e^{2\alpha i} = -7 - 24i$ ;

(g)  $125e^{3\pi i} = -117 - 44i$ .

21.  $P = 10e^{-(\pi/3)i} = 5 - 5\sqrt{3}i$ ;  $\bar{P} = 10e^{(\pi/3)i} = 5 + 5\sqrt{3}i$ ;  
 $-P = 10e^{(2\pi/3)i} = -5 + 5\sqrt{3}i$ ;  $-\bar{P} = 10e^{-(2\pi/3)i} = -5 - 5\sqrt{3}i$ .
23. (a)  $12 + 5i$ ; (b)  $33 + 56i$ ; (c)  $-12 - 5i$ ; (d)  $7 + 24i$ ; (e)  $63 + 16i$ .
25.  $-134217728 + 134217728\sqrt{3}i$ .
27. (c)  $2\sqrt{117}$ .

### Exercises for Chapter II Sections 6 and 7

3.  $3.02e^{0.1499i}$      $3.02e^{1.4065i}$      $3.02e^{2.6631i}$   
 $3.02e^{-2.3634i}$      $3.02e^{-1.1068i}$
9. (a)  $5.95e^{-2.5156i} = -4.82 - 3.49i$ ; (b)  $0.64e^{-3.1229i} = -0.64 - 0.01i$ ;  
(c)  $34.79e^{0.8303i} = 23.47 + 25.68i$ ; (d)  $69.48e^{2.5099i} = -56.07 + 41.03i$ .
11. 10.55
13. (a)  $53^\circ$ ;  $344^\circ$ ;  $200^\circ$ ;  $102^\circ$ .  
(b) 16.5 miles on a bearing of  $44^\circ$ .  
(c) 31.5 miles on a bearing of  $192^\circ$ .

### Exercises for Chapter III Section 1

1. (a)  $r = \sqrt{34}$ ;  $a = -5$ ;  $b = -3$ .

$$(b) \cos(\theta) = -\frac{5}{\sqrt{34}}; \quad \sin(\theta) = -\frac{3}{\sqrt{34}}; \quad \cot(\theta) = \frac{5}{3}$$

$$\sec(\theta) = -\frac{\sqrt{34}}{5}; \quad \csc(\theta) = -\frac{\sqrt{34}}{3}.$$

$$3. r = \sqrt{13}; \quad a = -3; \quad b = 2; \quad \cos(\beta) = -\frac{3}{\sqrt{13}}; \quad \tan(\beta) = -\frac{2}{3}.$$

$$5. (a) \frac{\sqrt{3}-1}{2\sqrt{2}}; \quad (b) \frac{\sqrt{3}+1}{2\sqrt{2}}; \quad (c) \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

$$7. (a) -\frac{29}{17\sqrt{13}}; \quad (b) -12/13; \quad (c) -3/2; \quad (d) -15/17$$

11. HINT: Make use of Euler formula and its conjugate.

### Exercises for Chapter III Section 2

$$3. \text{Arcsin}(1/2) = \frac{\pi}{6}; \quad \frac{\pi}{6} + 2\pi n \text{ and } \frac{5\pi}{6} + 2\pi n \text{ for any integer } n.$$

$$5. (a) \frac{\pi}{2}; \quad (b) \frac{\pi}{4}; \quad (c) \frac{\pi}{3}.$$

### Exercises for Chapter III Section 3

$$3. \alpha = 75^\circ; \quad x = \frac{8(\sqrt{3}+1)}{\sqrt{2}}.$$

### Exercises for Chapter III Section 4

$$1. \cos(38.4^\circ) = 0.7837; \quad \sin(38.4^\circ) = 0.6211; \quad \tan(38.4^\circ) = 0.7926;$$

$$\sec(38.4^\circ) = 1.2760; \quad \csc(38.4^\circ) = 1.6099; \quad \cot(38.4^\circ) = 1.2617.$$

3.  $\cos(-0.447) = 0.9017$ ;     $\sin(-0.447) = -0.4323$ ;     $\tan(-0.447) = -0.4794$ ;  
     $\sec(-0.447) = 1.1090$ ;     $\csc(-0.447) = -2.3134$ ;     $\cot(-0.447) = -2.0861$ .

5. 1.150

13. Distance is 342.3 m and the angle is  $27.5^\circ$ .

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