Chapter III

Trigonometry Using Complex Numbers

1. The Trigonometric Functions

Let θ be any real number. To obtain the trigonometric functions of θ one finds $r > 0$, a, and b such that $re^{\theta i} = a + bi$ and then uses the definitions:

$$\text{(D)} \qquad \begin{array}{lll} \cos(\theta) = \dfrac{a}{r}, & \sin(\theta) = \dfrac{b}{r}, & \tan(\theta) = \dfrac{b}{a}, \\[2ex] \sec(\theta) = \dfrac{r}{a}, & \csc(\theta) = \dfrac{r}{b}, & \cot(\theta) = \dfrac{a}{b}. \end{array}$$

The above three letter functions are abbreviations for cosine, sine, tangent, secant, cosecant, and cotangent, respectively. We note that the requirement $r > 0$ insures that cosine and sine are defined for all real numbers θ, but the other four functions will be undefined for values of θ which cause a zero in the denominator. [See Problem 10 below.] When any of these functions is defined, however, Problem 37, Exercises for Chapter 2 Sections 3, 4, and 5 shows that the definition is not ambiguous; that is, the functions are ***well defined***. For example, since $\sqrt{2}e^{(3\pi/4)i} = -1 + i$, we have

$$\cos(3\pi/4) = \frac{a}{r} = \frac{-1}{\sqrt{2}}, \ \sin(3\pi/4) = \frac{b}{r} = \frac{1}{\sqrt{2}}, \ \tan(3\pi/4) = \frac{b}{a} = \frac{1}{-1} = -1,$$

and the other three functions are the reciprocals of these. The same results would be obtained from $5\sqrt{2}e^{(3\pi/4)i} = -5 + 5i$ or any other nonzero complex number with $3\pi/4$ as argument.

The definitions show that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \ \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}, \ \sec(\theta) = \frac{1}{\cos(\theta)}, \ \text{and} \ \csc(\theta) = \frac{1}{\sin(\theta)}.$$

The definitions also imply that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Then

$$re^{\theta i} = a + bi = r\cos(\theta) + r\sin(\theta)i = r(\cos(\theta) + i\sin(\theta)).$$

For complex numbers of absolute value 1, i.e., for $r = 1$, this becomes the ***Euler Formula***

$$e^{\theta i} = \cos(\theta) + i\sin(\theta).$$

Replacing θ with $-\theta$, we get $e^{-\theta i} = \cos(-\theta) + i\sin(-\theta)$. Taking conjugates of each side of the Euler Formula, we get $e^{-\theta i} = \cos(\theta) - i\sin(\theta)$. These equations show that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Example 1. ***Double angle formulas*** for cosine and sine.

We use the Euler Formula to obtain $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ as follows:

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= e^{2\theta i} = \left(e^{\theta i}\right)^2 \\ &= \left(\cos(\theta) + i\sin(\theta)\right)^2 \\ &= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\left(2\sin(\theta)\cdot\cos(\theta)\right).\end{aligned}$$

Equating the real and imaginary parts on each side of the equation, we have

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

and

$$\sin(2\theta) = 2\sin(\theta)\cdot\cos(\theta).$$

These are the double angle formulas for cosine and sine respectively.

Similarly one can derive half angle formulas for $\cos(\theta/2)$ and $\sin(\theta/2)$ in terms of $\cos(\theta)$ using the fact that $e^{(\theta/2)i}$ is one of the two square roots of $e^{\theta i}$. [See Problems 8 and 18 below.]

Example 2. ***Addition Formulas*** for cosine and sine.

We use the Euler Formula to express $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ in terms of $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\beta)$, and $\sin(\beta)$ as follows:

$$e^{(\alpha + \beta)i} = e^{\alpha i}e^{\beta i} = \left(\cos(\alpha) + i\sin(\alpha)\right)\left(\cos(\beta) + i\sin(\beta)\right);$$

$$\begin{aligned}\cos(\alpha + \beta) + i\sin(\alpha + \beta) &= \left(\cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)\right) \\ &\quad + i\left(\sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta)\right).\end{aligned}$$

If we now equate the real and imaginary parts on each side of the equation, we get

$$\cos(\alpha + \beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)$$

and

$$\sin(\alpha + \beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta).$$

These are the addition formulas for the cosine and the sine, respectively. A symbolic aid for remembering these formulas is

$$(C + iS)(c + is) = (Cc - Ss) + i(Sc + Cs).$$

Example 3. ***Subtraction formula*** for tangent.

We seek $\tan(\alpha - \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$. First we note that

$$\begin{aligned} e^{\theta i}\sec(\theta) &= \big(\cos(\theta) + i\sin(\theta)\big)\sec(\theta) \\ &= \cos(\theta)\cdot\sec(\theta) + i\sin(\theta)\cdot\sec(\theta) \\ &= \cos(\theta)\cdot\frac{1}{\cos(\theta)} + i\sin(\theta)\cdot\frac{1}{\cos(\theta)} \\ &= 1 + i\tan(\theta). \end{aligned}$$

What we need here is only that there is a real number k such that $ke^{\theta i} = 1 + i\tan(\theta)$. Taking conjugates, we have $ke^{-\theta i} = 1 - i\tan(\theta)$. Thus

$$\begin{aligned} re^{\alpha i}se^{-\beta i} &= \big(1 + i\tan(\alpha)\big)\big(1 - i\tan(\beta)\big); \\ rse^{(\alpha-\beta)i} &= \big(1 + \tan(\alpha)\cdot\tan(\beta)\big) + i\big(\tan(\alpha) - \tan(\beta)\big). \end{aligned}$$

It now follows from the definition of the tangent that

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\cdot\tan(\beta)}.$$

This is the subtraction formula for the tangent.

Exercises for Chapter III Section 1

Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. Given that $\tan(\theta) = 3/5$ and $\pi < \theta < 3\pi/2$:

   (a) Find r, a, b such that $re^{\theta i} = a + bi$ with the given angle θ.

   (b) Find the other 5 trigonometric functions of θ.

2. Given that $\cos(\phi) = 5/7$ and $-\pi < \phi < 0$, find a complex number in both polar and rectangular forms having ϕ as its argument and then find the other five trigonometric functions of ϕ.

3. Let β be an angle with $\sin(\beta) = \dfrac{2}{\sqrt{13}}$ and $\pi/2 < \beta < \pi$. Find *r, a, b* such that $re^{\beta i} = a + bi$ with the given β and then find $\cos(\beta)$ and $\tan(\beta)$.

4. Let $\tan(\alpha) = 8/15$ with α acute. Find a complex number in both polar and rectangular forms having α as its argument and then find $\cos(\alpha)$ and $\sin(\alpha)$.

5. Use the result of Problem 16 (b), Exercises for Chapter II Sections 3, 4, and 5 and definition (D) to find each of the following:

   (a) $\sin(15^o)$ (b) $\cos(15^o)$ (c) $\tan(15^o)$.

6. Use the result of Problem 16 (a), Exercises for Chapter II Sections 3, 4, and 5, the results of Example 3 in Section 5 of Chapter II and definition (D) to find each of the following:

   (a) $\sin(105^o)$; (b) $\cos(105^o)$; (c) $\tan(105^o)$;

   (d) $\sin(75^o)$; (e) $\cos(75^o)$; (f) $\tan(75^o)$.

7. Let α and β be as in Problems 3 and 4 above. Use **operations on complex numbers** and the results of Problems 3 and 4 to find the following: [Do **not** use subtraction formulas, double angle formulas, etc.]

   (a) $\cos(\alpha - \beta)$, (b) $\sin(2\beta)$, (c) $\tan\left(\dfrac{\pi}{2} - \beta\right)$, (d) $\cos(\pi - \alpha)$.

8. Given that $re^{\phi i} = -6 + 7i$, use square roots of complex numbers [See Problem 29, Exercises for Chapter II Sections 3, 4, and 5.] to find all possibilities for:

   (a) $\sin(\phi/2)$, (b) $\tan(\phi/2)$, (c) $\cos(\phi/2)$.

9. Use operations on complex numbers (not sum and difference formulas) to verify each of the following:

   (a) $\sin(\alpha + \dfrac{\pi}{2}) = \cos(\alpha)$; (b) $\cos(\alpha + \dfrac{\pi}{2}) = -\sin(\alpha)$;

   (c) $\sin(\alpha - \dfrac{\pi}{2}) = -\cos(\alpha)$; (d) $\cos(\alpha - \dfrac{\pi}{2}) = \sin(\alpha)$;

10. It is clear from definitions (D) that there are values of θ for which some of the trigonometric functions are not defined because the denominator of the defining fraction will be zero.

(a) Characterize all the values of θ for which the cosecant and cotangent are not defined.

(b) Characterize all the values of θ for which the tangent and secant are not defined.

11. Prove the ***Pythagorean Identity***; $\sin^2(\theta) + \cos^2(\theta) = 1$ for all angles θ.

12. Let $e^{\theta i} = c + is$. Find all six trigonometric functions of θ in terms of c and s.

13. Let $e^{\theta i} = c + is$. Find the six trigonometric functions of θ in terms of c for (a) $0 \le \theta \le \pi$ and (b) $\pi \le \theta \le 2\pi$. [HINT: Use the results of problems 11 and 12.]

14. Let $e^{\theta i} = c + is$. Express $e^{2\theta i}$, $e^{3\theta i}$, $e^{4\theta i}$, and $e^{5\theta i}$ in terms of c and s.

15. Use Problem 14 to express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$ for $n = 1, 2, 3, 4, 5$.

16. Express $\cos(n\theta)$ and $\dfrac{\sin(n\theta)}{\sin(\theta)}$ in terms of $\cos(\theta)$ for $n = 1, 2, 3, 4, 5$.

17. Derive the addition, subtraction, and double angle formulas for the cosine, sine, tangent, and cotangent using complex numbers. (Some of these are in Examples 1, 2, and 3.)

18. (a) Derive the half angle formulas $\cos(\alpha/2) = \pm\sqrt{\dfrac{1 + \cos(\alpha)}{2}}$ and

$\sin(\alpha/2) = \pm\sqrt{\dfrac{1 - \cos(\alpha)}{2}}$ using square roots of complex numbers. Explain choice of $\pm$ sign. [See Problem 31, Exercises for Chapter II Sections 3, 4, and 5.]

(b) Use part (a) to show that $\cos^2(\beta) = \dfrac{1+\cos(2\beta)}{2}$ and $\sin^2(\beta) = \dfrac{1-\cos(2\beta)}{2}$.

(c) Use Problem 28, Exercises for Chapter II Sections 3, 4, and 5 to show that

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)} = \frac{1 - \cos(\theta)}{\sin(\theta)}.$$

19. (a) Complete the following table: [HINT: See Problem 3, Exercises for Chapter II Sections 3, 4, and 5. and use $\sin(-\theta) = -\sin(\theta)$.]

x	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(x)$					0	1/2			

(b) Explain why $\sin(x + \pi) = -\sin(x)$ and $\sin(x + 2\pi) = \sin(x)$.

(c) Use parts (a) and (b) to tabulate $y = \sin(x)$ for $\pi/2 \le x \le 3\pi/2$.

(d) Graph $y = \sin(x)$ for $-\pi \le x \le 2\pi$.

20. (a) Graph $y = \cos(x)$ for $-\pi \le x \le 2\pi$.

(b) Graph $y = \tan(x)$ for $-\pi/2 < x < \pi/2$ and $\pi/2 < x < 3\pi/2$.

21. Use the formula $\csc(x) = 1/\sin(x)$ and Problem 19 to graph $y = \csc(x)$ for $0 < x < \pi$ and $\pi < x < 2\pi$.

22. Graph:

(a) $y = \sec(x)$ for $-\pi/2 < x < \pi/2$ and $-3\pi/2 < x < -\pi/2$.

(b) $y = \cot(x)$ for $0 < x < \pi$ and $\pi < x < 2\pi$.

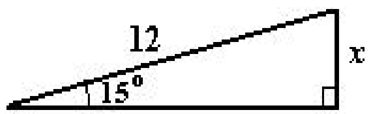
23. Consider a right triangle with α one of its acute angles. Let *hyp* be the length of the hypotenuse, *adj* the length of the side adjacent to α, and *opp* the length of the side opposite α. Verify each of the following:

(a) $\cos(\alpha) = \dfrac{adj}{hyp}$, (b) $\sin(\alpha) = \dfrac{opp}{hyp}$, (c) $\tan(\alpha) = \dfrac{opp}{adj}$,

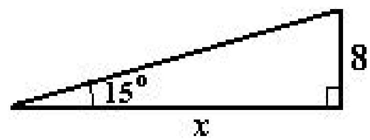
(d) $\sec(\alpha) = \dfrac{hyp}{adj}$, (e) $\csc(\alpha) = \dfrac{hyp}{opp}$, (f) $\cot(\alpha) = \dfrac{adj}{opp}$.

24. Use the results of problems 5 and 23 above to find x in each of the following.

(a)



(b)



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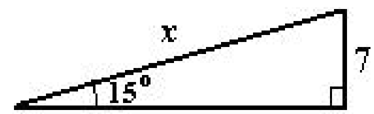

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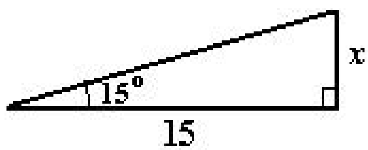
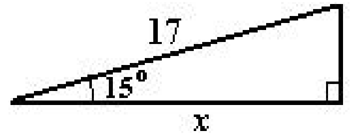


(e)



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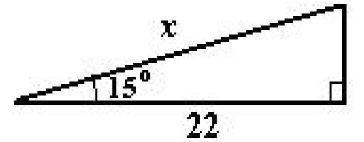
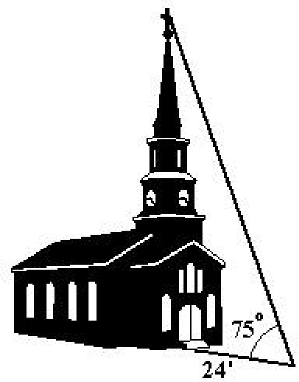


25. There is concern that a church steeple is too high for a new airport runway which is to be extended near by. An airport official stands at a point 24 feet from the point directly below the top of the steeple. From that location the angle of elevation to the top of the steeple is 75°. How tall is the steeple? [HINT: See problem 24 above.]



2. The Inverse Trigonometric Functions.

If f and g are functions such that f(a) = b if and only if g(b) = a, then f and g are ***inverse functions*** of each other. For example, the functions f and g with $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverse functions of one another since $b = a^3$ if and only if $a = \sqrt[3]{b}$. [Note that the inverse function of $f(x) = x^3$ is $g(x) = \sqrt[3]{x}$ while the additive inverse or negative is $-f(x) = -x^3$ and the multiplicative inverse or reciprocal is $1/f(x) = 1/x^3$.]

A function f that has the same value b for two different numbers a and c in its domain can not have an inverse function g since f(a) = b = f(c), with f and g inverses of each other, implies a = g(b) = c. Each of the trigonometric functions sine, cosine, tangent, cotangent, secant, and cosecant repeats its values in intervals of 2π and hence does not have an inverse function.

However the trigonometric functions with suitably restricted domains have inverses. For example, as x increases from $-\pi/2$ to $\pi/2$, $\sin(x)$ increases steadily from -1 to 1; hence the sine function with domain restricted to the interval $-\pi/2 \le x \le \pi/2$ does not repeat values and so has an inverse function. We use Sine (abbreviated Sin) to denote the sine function with domain the closed interval $[-\pi/2,\ \pi/2]$ and designate its inverse as Arcsine (Arcsin). The domain of the Arcsine function is [-1, 1] and its range is $[-\pi/2,\ \pi/2]$.

Similarly, the tangent on the open interval $-\pi/2 < x < \pi/2$ takes on all real values once and only once and so has an inverse. We designate the tangent function with domain restricted to the open interval $(-\pi/2,\ \pi/2)$ as Tangent (Tan) and its inverse as Arctangent (Arctan).

As x varies from 0 to π, cos(x) decreases steadily from 1 to -1. Therefore we use Cosine (Cos) to designate the restriction of the cosine function to the domain $[0,\pi]$; its inverse is written as Arccosine (Arccos).

The essential facts about these three inverse trigonometric functions are:

y = Arcsin(x) means that x = sin(y) and $-\pi/2 \leq y \leq \pi/2$,
y = Arctan(x) means that x = tan(y) and $-\pi/2 < y < \pi/2$,
y = Arccos(x) means that x = cos(y) and $0 \leq y \leq \pi$.

Frequently Arcsin(x), Arctan(x), and Arccos(x) are written $\sin^{-1}x$, $\tan^{-1}x$, and $\cos^{-1}x$ respectively. One should be prepared for this bad notation in the literature and not allow it to make one confuse an inverse function with a reciprocal. For example, $\sin^{-1}x$ is Arcsin(x) but is **not** $(\sin(x))^{-1} = \csc(x)$.

Exercises for Chapter III Section 2

Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. What might be the motivation for choosing the interval $[0,\pi]$ as the domain of the Cosine function?

2. Give the domain and range of: (a) Arctan(x); (b) Arccos(x).

3. Find Arcsin(½) and four other real numbers x such that sin(x) = ½.

4. Find $\text{Arctan}\left(\sqrt{3}\right)$ and four other real numbers x such that $\tan(x) = \sqrt{3}$.

5. Find: (a) Arcsin(1); (b) Arctan(1); (c) Arccos(½).

6. Find: (a) $\text{Arcsin}\left(\frac{-\sqrt{2}}{2}\right)$; (b) $\text{Arctan}\left(-\sqrt{3}\right)$; (c) Arccos(-1).

7. (a) Graph both y = Cos(x) and y = Arccos(x) on the same axes.

(b) Are these graphs symmetric to each other with respect to some line?

8. Graph; (a) y = Arcsin(x); (b) y = Arctan(x).

3. Solving Triangles

There are three important congruence theorems from geometry that are usually abbreviated as ASA, SAS, and SSS. ASA, for example, tells us that if two angles and the included side of one triangle are equal to the respective two angles and included side of another triangle, the two triangles are congruent. Similar statements apply to the other two abbreviations. What these theorems tell us, is that if the right three parts of a triangle are known, the other three parts are fixed and they should be able to be found. Having partial information about a triangle and using it to find the rest of the information about the triangle is referred to as ***solving the triangle***. The procedure for solving right triangles is demonstrated in Problem 24 of Exercises for Chapter 3 Section 1. This sections introduces two important laws of trigonometry that can be used for solving any triangle for which appropriate information is known.

Law of Sines

Consider the triangles in Figure 1a and Figure 1b. Since every triangle has at least two acute angles, we can assume we have picked one of them to call α. Since $\sin(\alpha) = \frac{h}{b}$, (see Problem 23, Exercises for Chapter 3 Section 1.) $h = b\sin(\alpha)$. If β is acute as in Figure 1a, then $\sin(\beta) = \frac{h}{a}$. If β is obtuse as in Figure 1b, then $\beta' = 180^o - \beta$ and $\sin(\beta) = \sin(180^o - \beta) = \sin(\beta') = \frac{h}{a}$. In either case, $h = a\sin(\beta)$. Equating these values of h, we have $b\sin(\alpha) = a\sin(\beta)$ or

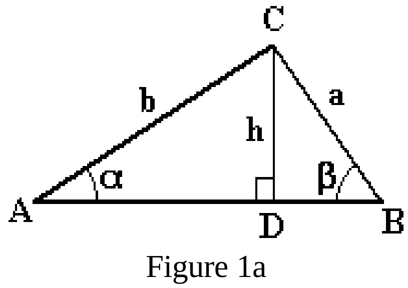


Figure 1a

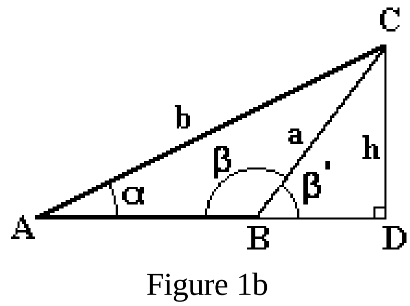


Figure 1b

$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b}$. Similarly, if γ is the angle at C and c is its opposite side, we can show that $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$.

We are now prepared to state the ***Law of Sines***: If a triangle has angles α, β, and γ with opposite sides a, b, and c respectively, (See Figure 1c) then

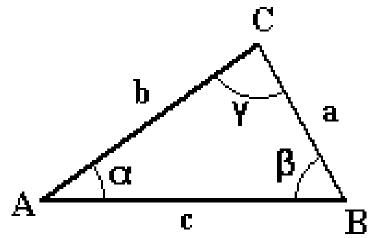


Figure 1c

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

All of the examples and problems which follow are based on triangles labeled as in Figure 1c.

Example 1. If $\alpha = 45^\circ, \beta = 60^\circ$, and $c = 8$, find angle C and sides a and b. (Given ASA.)

Solution: First, $\gamma = 180^\circ - (45^\circ + 60^\circ) = 75^\circ$. From Problems 6 and 19 of Exercises for Chapter III Section 1 we have $\sin(\alpha) = \frac{1}{2}\sqrt{2}$, $\sin(\beta) = \frac{1}{2}\sqrt{3}$, and $\sin(\gamma) = \frac{\sqrt{2}}{4}\left(1 + \sqrt{3}\right)$. Now, from the Law of Sines, $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$, thus $a = \frac{c\sin(\alpha)}{\sin(\gamma)} = \frac{8\left(\frac{1}{2}\sqrt{2}\right)}{\frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)} = 8\left(\sqrt{3} - 1\right)$.

Similarly, $b = 4\sqrt{6}\left(\sqrt{3} - 1\right)$.

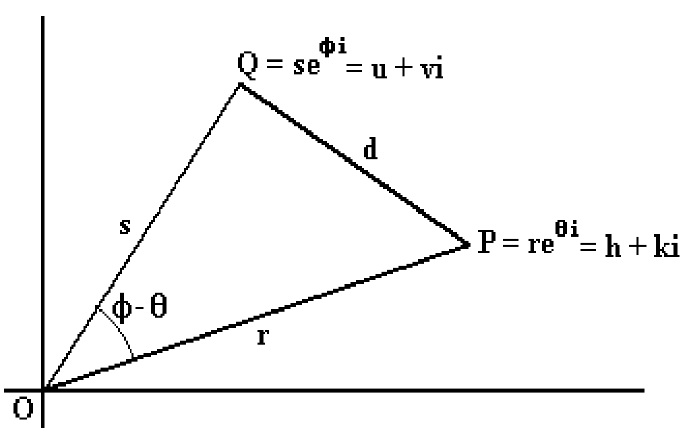


Figure 2

Law of Cosines

Consider the complex numbers P and Q as shown in Figure 2. By Problem 33 of Exercises for Chapter II Sections 3, 4, and 5, we see that d, the distance between P and Q, is

$$d = |Q - P| = \sqrt{(u - h)^2 + (v - k)^2}.$$

But $h = r\cos(\theta)$, $k = r\sin(\theta)$, $u = s\cos(\phi)$, $v = s\sin(\phi)$. Substituting these into the equation for d, squaring both sides, expanding, and collecting like terms, gives us

$$
\begin{aligned}
d^2 &= (s\cos(\phi) - r\cos(\theta))^2 + (s\sin(\phi) - r\sin(\theta))^2 \\
&= s^2(\cos^2(\phi) + \sin^2(\phi)) + r^2(\cos^2(\theta) + \sin^2(\theta)) - 2sr(\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)).
\end{aligned}
$$

Using the Pythagorean Identity [See Problem 11, Exercise for Chapter III Section 1.] and the subtraction formula for the cosine [See Problem 17, Exercise for Chapter III Section 1.] this becomes

$$d^2 = s^2 + r^2 - 2sr\cos(\phi - \theta).$$

If ΔOPQ is now relabeled as in Figure 3c, we get the three forms of the ***Law of Cosines***:

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos(\alpha) \\
b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\
c^2 &= a^2 + b^2 - 2ab\cos(\gamma)
\end{aligned}
$$

depending on whether angle A, B or C, respectively is placed at the origin.

Example 2. If $\gamma = 15^\circ$, $a = 5\sqrt{6}$, and b = 10, find side c and angles A, and B. (Given SAS.)

Solution: From Problem 5 of Exercises for Chapter 3 Section 1, $\cos(15^\circ) = \frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)$.

Then, from the Law of Cosines,

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos(\gamma) = 150 + 100 - 50\sqrt{3}\left(1+\sqrt{3}\right) \\
&= 100 - 50\sqrt{3} = 25\left(4 - 2\sqrt{3}\right) = 5^2\left(\sqrt{3}-1\right)^2.
\end{aligned}
$$

Thus, $c = 5\left(\sqrt{3}-1\right)$. From the Law of Sines $\frac{\sin(\beta)}{b} = \frac{\sin(15^\circ)}{c}$. Since

$\sin(15^\circ) = \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$,

$$
\begin{aligned}
\beta &= \operatorname{Arcsin}\left(\frac{b\sin(\gamma)}{c}\right) = \operatorname{Arcsin}\left(\frac{10\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)}{5\left(\sqrt{3}-1\right)}\right) \\
&= \operatorname{Arcsin}\left(\frac{1}{2}\sqrt{2}\right) = 45^\circ.
\end{aligned}
$$

Finally, $\alpha = 180^\circ - (15^\circ + 45^\circ) = 120^\circ$.

Exercises for Chapter III Section 3

Do NOT use a calculator for any of these problems, give only exact answers. (See Preface)

1. (a) Reconsider Examples 2. After finding c with the Law of Cosines, use the Law of Sines and Arcsin to find α, then use subtraction to find β.

   (b) Why are the answers different from the correct answers given in Example 2?

   (c) What must be done to insure getting the correct answers in this type of problems?

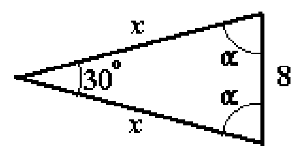
2. Solve each of the following triangles (See Figure 1c) from the given information if possible. The results of Problems 5 and 6 of Exercises for Chapter III Section 1 may be helpful here.

   (a) $\alpha = 45°, \beta = 30°$, and $c = 12$.

   (b) $\beta = 15°$, $a = 10\sqrt{2}\left(3 - \sqrt{3}\right)$, and $c = 20$. HINT: $a < c$ in this case.

   (c) $a = 3\left(\sqrt{3} - 1\right), b = 3\sqrt{2}$, and $c = 6$. HINT: Use the Law of Cosines first to find one of the angles.

3. Solve the isosceles triangle in the figure to the right.



4. Show that if you are given two angles and a side which is not the included side, (AAS or SAA), the triangle is still determined. This is usually a corollary to the ASA theorem.

5. Why is AAA not a congruence theorem?

6. Show that SSA does not determine a triangle by finding two distinct triangles which satisfy $\alpha = 30°$, $a = 5\sqrt{2}\left(\sqrt{3} - 1\right)$, and $b = 10$.

4. Trigonometry on the Calculator.

We note that the calculator has keys labeled SIN, COS, and TAN. These are, as might be expected, the keys for the sine, the cosine, and the tangent functions respectively. If the real number x is on level 1 of the stack, pressing SIN will give $\sin(x)$, TAN will give $\tan(x)$, and COS will give $\cos(x)$. The real number x will be interpreted as degrees or radians according to the angle mode setting. For example if the number 30 is on level 1 of the stack and the calculator is in degree mode, pressing SIN gives the result .5 as expected. If, however, the calculator is in radian mode with 30 on level 1 of the stack, the result of pressing SIN is -.988031624093, which is the sine of 30 radians.

There are no keys for the cosecant, secant, or cotangent functions. Since these functions are the reciprocals of the sine, cosine, and tangent respectively, they can be obtained with the keys we have and the 1/x key. The trigonometric functions are on pages 3 - 6 of ***UG***.

Calculator Example 3.4.1

Find $\sec(\pi/3)$.

Solution: With the calculator in radian mode key in LS π 3 $\div$ COS 1/x. We see the result 2.00000000001. The answer, of course, should be 2, but we are again seeing an example of the round off errors caused by using a finite machine to approximate computations with real numbers.

We see that the left shift functions for SIN, COS, and TAN are labeled ASIN, ACOS, and ATAN, respectively. These are, respectively, the Arcsine, Arccosine, and Arctangent functions. These functions are not inverses of each other since, for example, SIN is the sine function, not the Sine function.

Calculator Example 3.4.2

Find $\tan(3\pi/4)$; then take the Arctangent of the result.

Solution: With the calculator in radian mode key in 3 LS π $\times$ 4 $\div$ TAN, and we see the expected answer -1. Now key in LS ATAN and we get the result -.785398163397, which is the decimal approximation for $-\pi/4$. This should not be surprising since $3\pi/4$ is not in the domain of Tan and so $\text{Arctan}(\tan(\theta))$ is not necessarily θ for this angle.

Before starting the next example, you may want to review the plotting instruction in Chapter 12 of ***UG***.

Calculator Example 3.4.3

Graph $y = \sin(x)$ for $-180° \le x \le 360°$ on the calculator.

Solution: The first step is to insure your calculator will react as indicated by these instructions. If you have variables called X, EQ, and/or PPAR in your variable list, purge them. (See page 2-32 and 2-33 of ***UG***.)

Set the calculator to degree mode. Now key in LS(hold) 2D/3D to get into the PLOT SETUP dialog box. (**REMEMBER** - If your calculator is in RPN mode, you must **HOLD** LS while pressing 2D/3D or WIN to get into those dialog boxes.) Press DA ' SIN F1-X ENTER to enter 'SIN(X)' into the EQ: box. We now want to set the horizontal ticks to 30 degrees and the vertical ticks to .2 units. Press DA 30 ENTER .2 ENTER F2-CHK. Your screen should now look like Figure 3a.

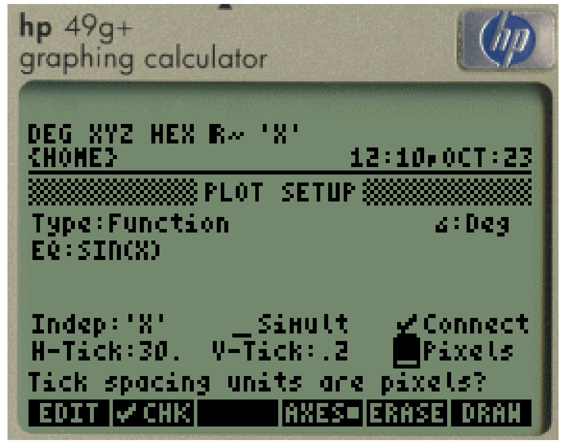


Figure 3a

Now key in NXT F6-OK LS(hold) WIN to get into the PLOT WINDOW dialog box. Now key 180 +/- ENTER 360 ENTER to set the H-VIEW: boxes and F4-AUTO to have the calculator select the best values for V-VIEW: boxes. The dialog box should now look like Figure 3b.

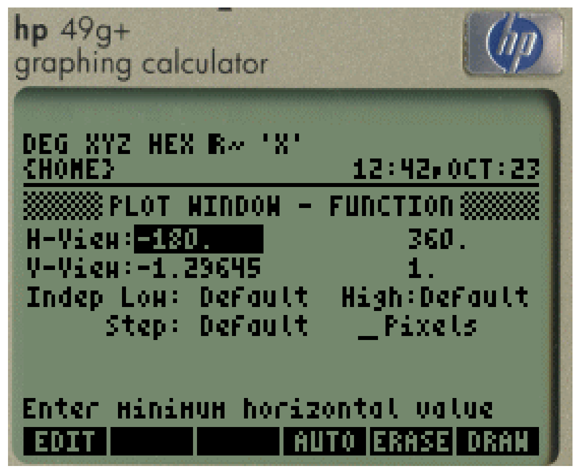


Figure 3b

Now press F5-ERASE F6-DRAW to draw the graph which we see in Figure 3c.

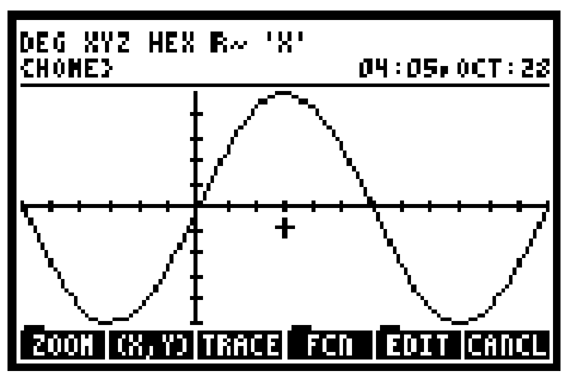


Figure 3c

Calculator Example 3.4.4

Graph y = Cos(x) and y = Arccos(x) on the same axes. See Problem 7, Exercises for Chapter III Section 2.

Solution: Assuming your calculator is as you left it from the previous example, press CANCEL or F6-CANCL to return to the PLOT WINDOW dialog box. Now key NXT F1-RESET DA F6-OK. This resets the plot parameters back to their default state. In this state one unit in both directions is 10 pixels, so geometric properties will not be distorted, but the two views are about twice as big as we need them. We will cut each of the fields in the two view areas in half, and will add .5 to the two V-VIEW fields to move the origin down that amount. To accomplish this, press F2-CALC 2 ÷ F6-OK to set the left field of H-VIEW. Now key NXT F2-CALC 2 ÷ F6-OK to set the right field of H-VIEW. For each of the two fields of V-VIEW key NXT F2-CALC 2 ÷ .5 + F6-OK.

Now press 0 ENTER so the graph will start at $x = 0$ and LS π to end the graph at $x = \pi$. Your screen should now look like Figure 4a.

Now press NXT F6-OK LS(hold) 2D/3D to get into the PLOT SETUP dialog box. Key RA +/- to change the angle mode to radians, then RA ‹ COS F1-X ENTER to change to the cosine function. Now key DA .5 ENTER .5 ENTER to change the ticks to .5 units in both directions. The screen should now look like Figure 4b.

Now press F5-ERASE F6-DRAW, and when the graph is complete, press the minus sign to remove the menu. You should see the graph shown in Figure 4c.

Press CANCEL to get back to the PLOT SETUP dialog box and change the EQ: field to ACOS(X). Now press LS(hold) WIN to get into the PLOT WINDOW and change Indep Low: to -1 and High: to 1 to plot the domain [-1, 1]. Press F6-DRAW **without** erasing. You should now see the graph shown in Figure 4d.

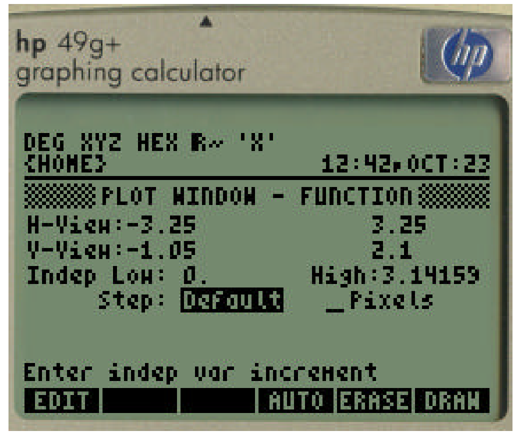


Figure 4a

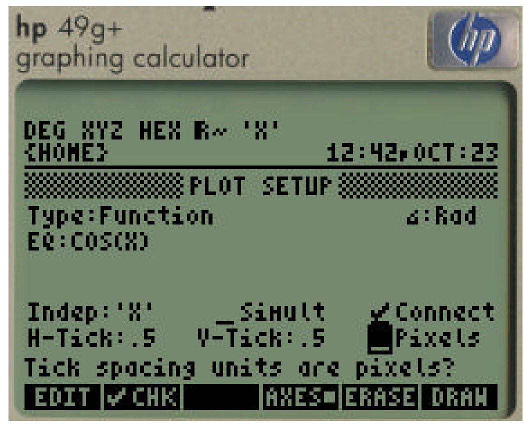


Figure 4b

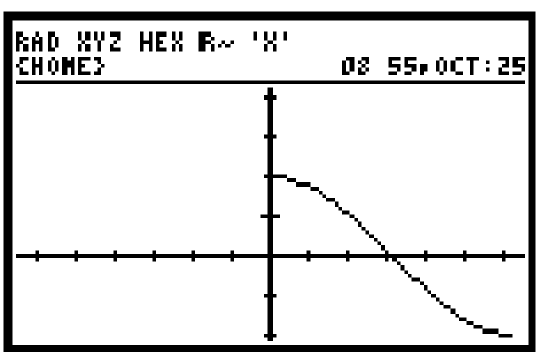


Figure 4c

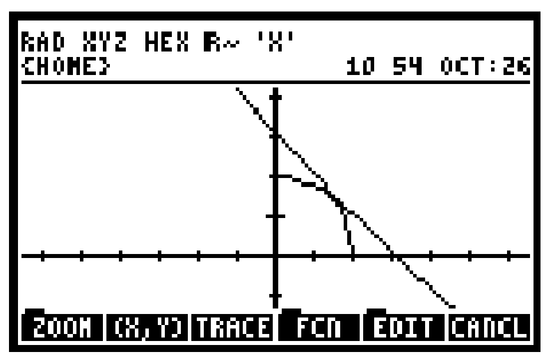


Figure 4d

Finally, let us add the line of symmetry for cos(x) and Arccos(x). Press F2-(X,Y) and use RA and UA to move the cursor until the coordinates at the bottom of the screen show X:2. Y:2. (NOTE: holding the arrow keys down will cause the cursor to move more quickly.) Now press × to put a mark on the screen as shown in Figure 4e. Next press + to get the menu back, then press F5-EDIT. Use DA and LA to move the cursor back to the origin, press F3-LINE, then $\times$, and you should see the graph shown in Figure 4f.

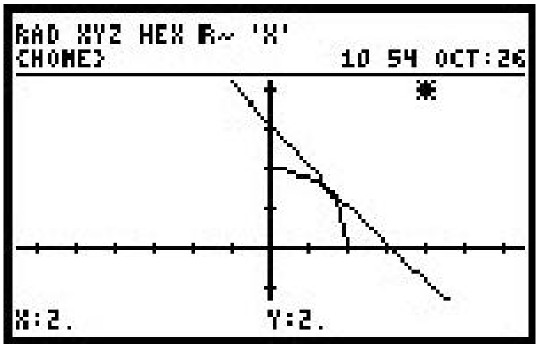


Figure 4e

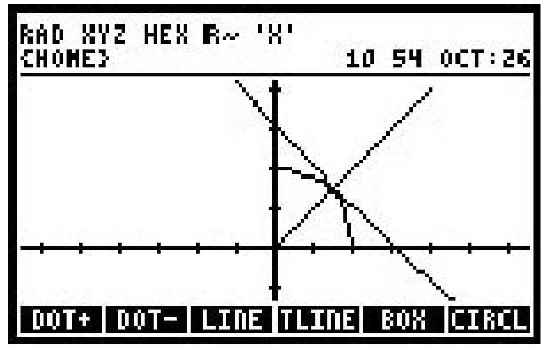


Figure 4f

Calculator Example 3.4.5

In ΔABC, $\alpha = 37.1^o$, $\beta = 58.3^o$, and $c = 47.26$. Solve the triangle. Find the angles to one decimal place and the lengths to two decimal places.

Solution: First $\gamma = 180^o - \alpha - \beta$, so on the calculator we key 180 ENTER 37.1 - 58.3 - and we see that $\gamma = 84.6°$. From the Law of Sines, $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$. We solve this for a and substitute the known quantities to get $a = \frac{47.26\sin(37.1^o)}{\sin(84.6^o)}$. With the calculator in degree mode and set to Fix 2, we key in 47.26 ENTER 37.1 SIN $\times$ 84.6 SIN $\div$ and obtain $a = 28.63$.
Similarly

$$b = \frac{c\sin(\beta)}{\sin(\gamma)} = \frac{47.26\sin(58.3°)}{\sin(84.6°)} = 40.39.$$

Calculator Example 3.4.6

Given that in ΔABC one has $\gamma = \pi/6$, $a = 22.16$, and $b = 43.26$, solve the triangle. Give angles to three decimal places and lengths to 2 decimal places.

Solution: We observe that the given information is of the form SAS, hence the triangle is determined. We first use the Law of Cosines to find

$$c = \sqrt{a^2 + b^2 - 2ab\cos(\gamma)}$$

$$= \sqrt{22.16^2 + 43.26^2 - 2 \cdot 22.16 \cdot 43.26\cos(\pi/6)}.$$

On the calculator (in radian mode) this is accomplished with 22.16 LS x^2 43.26 LS x^2 + 2 ENTER 22.16 × 43.26 × LS π 6 ÷ COS × − $\sqrt{x}$, which gives us c = 26.50. We now use the Law of Sines to find

$$\sin(\alpha) = \frac{a\sin(\gamma)}{c}$$

or

$$\alpha = \text{Arcsin}\left(\frac{22.16\sin(\pi/6)}{26.50}\right).$$

Before we continue, a reminder. The SWAP command can be found by TOOL F3-STACK F2-SWAP. It can also be found by LS RA, and if the calculator is in a state where RA would not make sense, the LS is not necessary.

Set the calculator to Fix 3. Now, assuming the value of c is still on the stack from the previous calculation, we proceed with 22.16 ENTER LS π 6 ÷ SIN × SWAP ÷ LS ASIN, which gives us $\alpha = .431$. (Notice that when we were ready to divide by c we took advantage of the fact that it was already on the stack and just used the SWAP command to put it in the right position for the division. Every time one can eliminate the need to key in a number, one has removed an opportunity to make an error.) Finally, we compute the last angle by subtracting the first two from π and get $\beta = 2.187$.

Exercises for Chapter III Section 4

In all of the following problems give trigonometric functions to 4 decimal places, find all angles in the same units as angles given in the problem with 1 decimal place for degrees and 3 decimal places for radians, and give all lengths with the same number of decimal places as lengths given in the problem.

1. Find the six trigonometric functions for $\alpha = 38.4°$.

2. Find the six trigonometric functions for $\beta = 7\pi/12$.

3. Find the six trigonometric functions for $\gamma = -0.447$ radians.

4. Give each of the following in degrees:

   (a) Arcsin(0.7739); (b) Arcsin(-0.7739); (c) Arccos(0.7739); (d) Arccos(-0.7739);

(e) Arctan(0.7739); (f) Arctan(-0.7739).

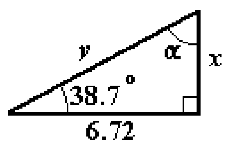
5. Find an angle in radians whose secant is 2.4483.

6. Graph cos(θ) for $-\pi/2 \le \theta \le 3\pi/2$ and $\cos(\theta - \pi/6)$ for $-\pi/3 \le \theta \le 5\pi/3$ on the same plot on the calculator.

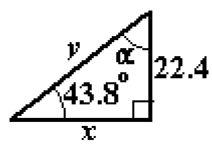
7. Do Problem 24 of Exercises for Chapter III Section 1 on the calculator and compare your answers to the exact answers.

8. Solve each of the following right triangles:

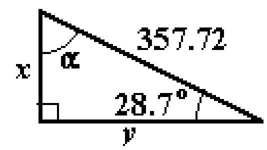
(a)



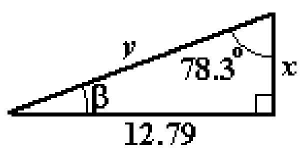
(b)



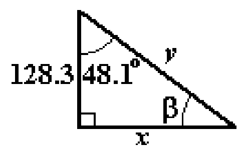
(c)



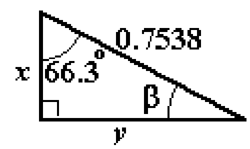
(d)



(e)

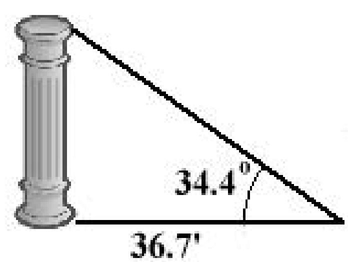


(f)



9. Do Problem 25 of Exercises for Chapter III Section 1 on the calculator and compare your answers to the exact answers.

10. An architect is interested in finding the height of the column shown on the right. She goes to a point 36.7 feet from the base of the column, from which point the angle of elevation to the top is 34.4°. How high is the column?



11. Use the calculator to solve Problem 2 of Exercises for Chapter III Section 3 and compare your answers to the exact solutions.

12. Solve each of the following triangles (See Figure 1c) from the given information if possible.

(a) $\alpha = 72.3°$, $\beta = 47.6°$, $b = 39.47$;

(b) $\beta = 5\pi/12$, $a = 2.917$, $c = 3.264$;

(c) $a = 472.6$, $b = 515.1$, $c = 497.7$, find angles in radians.

13. A contractor needs to run a sewer line from a new building at *A* to the waist treatment plant at *B*, but heavy underbrush makes it difficult to see directly from *A* to *B*. From *C*, however, he can clearly see both *A* and *B*. The distance *AC* is 218.6 meters, the distance *BC* is 179.4 meters, and the angle at *C* is 118.3 degrees. Find the distance *AB* and the angle α at *A*.