Chapter I

Preliminaries from Geometry

This chapter contains definitions, axioms, and theorems from geometry which are needed for what follows. They serve to insure that the reader and the authors have a common terminology for the material which is prerequisite to the study of trigonometry, so they are, for the most part, presented as facts without formalism or proof.

1. Rays and Segments

A point P on a (straight) line λ divides λ into two half-lines, each of which is a ***ray*** with P as its only endpoint. A ray extends infinitely in one direction. Let P and Q be distinct points and λ be the unique line passing through them. Then the ***ray PQ*** designates the ray with P as endpoint which passes through Q. Also, the ***segment PQ*** consists of P and Q and all the points between these endpoints on the line λ. By specifying P as the initial point and Q as the final point, the segment PQ becomes the "***directed segment***" $\vec{PQ}$.

Directed segments $\vec{AB}$ and $\vec{CD}$ have the same magnitude if the lengths of the segments are equal, as in Figure 1.

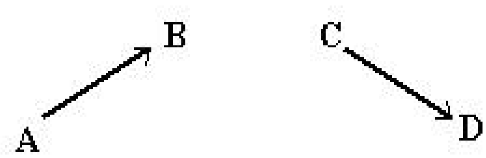


Figure 1

If $\vec{AB}$ and $\vec{CD}$ are on parallel lines, (or are on the same line), they may have the same direction, as in Figure 2a, or they may have opposite directions, as in Figure 2b.

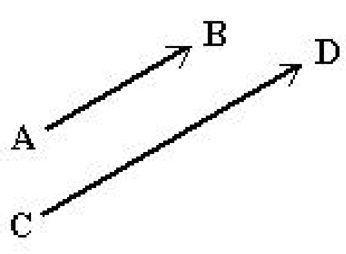


Figure 2a

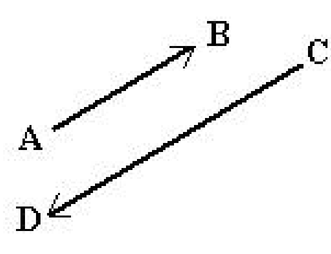


Figure 2b

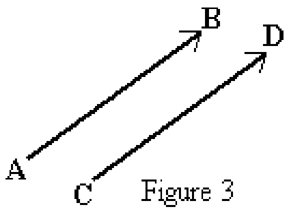


Figure 3

If directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same magnitude and direction as in Figure 3, this will be denoted by the notation

$$\overrightarrow{AB} \cong \overrightarrow{CD}.$$

2. Angles and Triangles

The angle formed by segments BA and BC (or rays BA and BC) is denoted by $\angle ABC$; the point B is its ***vertex***. If no other angle with vertex at B is under consideration, $\angle ABC$ may be shortened to just $\angle B$. Triangle ABC is denoted by ΔABC.

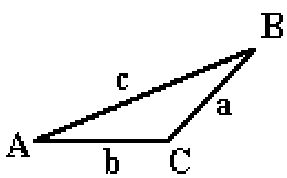


Figure 4a

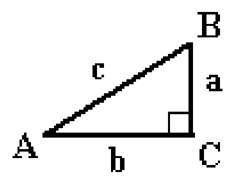


Figure 4b

The sum of the lengths of two sides of a triangle is always greater than the length of the third side. For example, $a + b > c$ in Figures 4a and 4b. Angles are measured either in degrees or radians. The degree measure of a right angle is 90^o and its radian measure is $\pi/2$. The sum of the degree measures of the angles of a triangle is 180^o and the sum of their radian measures is π. An angle of a triangle is ***acute***, ***right***, or ***obtuse*** depending on whether its degree measure is respectively, less than , equal to, or greater than 90^o. The notation $\angle A = 30^o$ means that the degree measure of $\angle A$ is 30^o and $\angle A = \pi/6$ means that the radian measure of $\angle A$ is $\pi/6$.

A triangle with one 90^o angle is called a right triangle. The side opposite the right angle is called the ***hypotenuse***. In Figure 4b, $\angle C = 90^o$ and side c is the hypotenuse. The famous Theorem of Pythagoras states that a triangle is a right triangle if and only if the square of the length of one side equals the sum of the squares of the lengths of the other two sides. (For example, $c^2 = a^2 + b^2$ in Figure 4b.) The main steps of a proof follow:

Let ΔABC have a right angle at C. Place a square of side c externally on the hypothenuse AB. Then place copies of ΔABC on the other three sides of the square. All together, we now have a big square whose sides have length $a + b$. See Figure 5.

Now we get

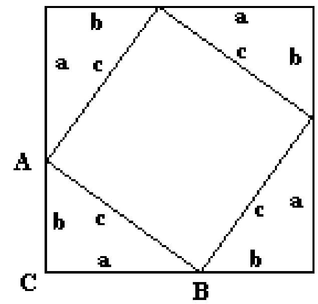


Figure 5

$$c^2 = (a+b)^2 - 4 \cdot \frac{1}{2}ab$$
$$c^2 = a^2 + 2ab + b^2 - 2ab$$
$$c^2 = a^2 + b^2$$

Calculator Example 1.2.1

When laying out the foundation of a new building, stakes are driven at the four corners. Measuring the proper distances between the stakes is relatively easy, but measuring the angles properly is much more difficult. After driving the stakes, the builder always measures the diagonal distance to make sure it satisfies the Theorem of Pythagoras thereby insuring that the angles are 90^o. Suppose that the points *A, B,* and *C* in Figure 4b are three of the four corner stakes of a rectangular foundation, *a* = 36 ft, and *b* = 40 ft. What must *c* be to insure $\angle C = 90^o$? Give your answer to the nearest 10th of an inch.

Solution: We will assume the calculator has already been set to Fix 1 display mode (see Preface). We will let the calculator keep track of the units for us, so the first sequence is to get to the appropriate units menu: RS UNITS F2-LENGTH. For each of the given lengths we will enter the value, attach the units, then square it: 36 F5-ft LS x^2 40 F5-ft LS x^2. We now have the squares of *a* and *b* on the stack. To find *c* we must add these, take the square root, then convert the result to inches: + $\sqrt{x}$ LS F6-in. We see the answer 645.8_in on the display. For complete instructions on the use of units see "Operations with Units" starting on page 3-17 of ***UG***.

3. Similar Triangles

By definition, ΔABC is similar to $\Delta A'B'C'$ if $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$. See Figure 6.

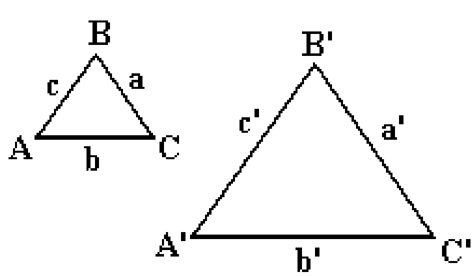


Figure 6

Each of the following conditions guarantees that ΔABC is similar to $\Delta A'B'C'$.

(a) Two angles of one triangle are equal respectively to the corresponding two angles of the other.

(b) Two sides of one triangle are proportional to the corresponding sides of the other triangle and the included angles are equal.

(c) The three sides of one triangle are proportional to the corresponding sides of the other.

If we are given that ΔABC is similar to $\Delta A'B'C'$ then

$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C', \text{ and } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

As stated in (a), (b), and (c), to prove that ΔABC is similar to $\Delta A'B'C'$, it suffices to show that $\angle A = \angle A'$ and $\angle B = \angle B'$ or to show that $\frac{a}{a'} = \frac{b}{b'}$ and $\angle C = \angle C'$ or to show that $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Calculator Example 1.3.1

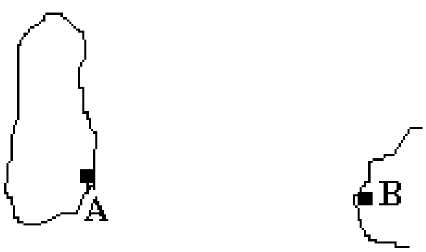


Figure 7a

While sailing around the island in Figure 7a you hit a submerged rock and tore a hole in your boat. You are now stuck on the island and wonder how far it is from point *A* on the island to point *B* on the mainland so you can decide if you can risk swimming it. You have a compass, a tape measure, and your trusty HP 49G+ calculator. How do you estimate the distance from *A* to *B*?

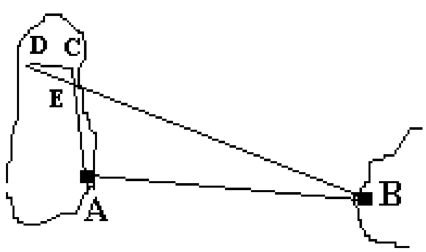


Figure 7b

Solution: From *A* use the compass to measure the bearing to *B*. Now walk along the beach to some point *C* making a line in the sand as you go. See Figure 7b. From *C* walk along the bearing 180º from the bearing you measured from *A* to *B* to some point *D* from which you can still see *B*. Put a marker at *D* and walk strait towards *B* until you get to the line *AC* and mark the point *E*.

Notice that since *AB* is parallel to *CD*, $\angle A = \angle C$. (See Section 5 below.) Also, $\angle AEB = \angle CED$, so ΔAEB is similar to ΔCED. We now have $\frac{AB}{CD} = \frac{AE}{CE}$. Using the tape measure you find that *CE* = 21 yd, *CD* = 67 yd, and *AE* = 113 yd. Now solve the previous proportion for *AB* and substitute the measurements, giving $AB = \frac{113 \cdot 67}{21}$. The sequence 113 ENTER 67 × 21 ÷ on the calculator shows the distance to be a bit over 360 yards.

4. Important Special Triangles

Two sides of a triangle have equal length if and only if the angles opposite them have equal measure. Such a triangle is called an ***isosceles*** triangle. It follows that the angles of an ***equilateral triangle*** (one having all sides equal) each measures 60^o.

Let ΔABC be equilateral with each side having 2 units as its length. Let M be the midpoint of side AC. See Figure 8. Then ΔAMB and ΔCMB are congruent right triangles, $\measuredangle ABM = 30^o$, $\measuredangle A = 60^o$, $\measuredangle AMB = 90^o$, side $AM = 1$, and the length h of side MB satisfies

$$h^2 + 1^2 = 2^2.$$

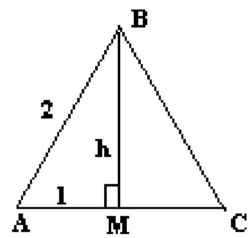


Figure 8

Thus $h = \sqrt{3}$ and the three sides of this 30^o, 60^o, 90^o triangle have lengths 1, $\sqrt{3}$, 2. If ΔEFG is any triangle with $\measuredangle E = 30°$, $\measuredangle F = 60°$, and $\measuredangle G = 90°$, then ΔEFG is similar to ΔBAM and its sides e, f, g must be proportional to 1, $\sqrt{3}$, 2. We can write this as $e : f : g = 1 : \sqrt{3} : 2$. It follows that the lengths of the sides of a 30^o, 60^o, 90^o triangle can be written as e, $\sqrt{3}\,e$, $2e$.

Now let ΔABC be an isosceles right triangle with $\measuredangle C = 90^o$, c as the hypotenuse and k as the length of each of the other two sides. Then $\measuredangle A = 45^o = \measuredangle B$ and $c^2 = k^2 + k^2$. It follows that $c = \sqrt{2}k$ and thus the sides of a 45^o, 45^o, 90^o triangle are of the form k, k, $\sqrt{2}k$.

Before computers, when drafting was done by hand, every draftsman had several of these special triangles of various sizes in his tool box. See figure 9a where ΔTUV is a 30^o, 60^o, 90^o triangle and ΔWXY is a 45^o, 45^o, 90^o triangle. With these, the draftsman could construct many angles of various sizes. In figure 9b we see an example of how these two triangles and addition of angles can be used to create a 75^o angle, and in figure 9c we see an example of how one of the triangles and subtraction is used to create an angle of 150^o.

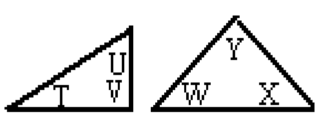


Figure 9a

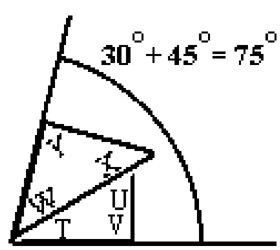


Figure 9b

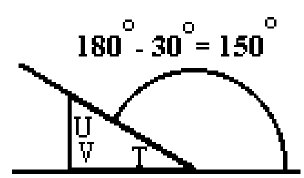


Figure 9c

5. Parallel Lines and Parallelograms

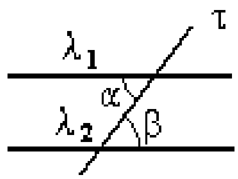


Figure 10

If two lines, λ_1 and λ_2, are cut by a transversal, τ, the lines are parallel if and only if the alternate interior angles α and β are equal. See Figure 10.

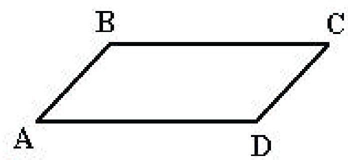


Figure 11

A parallelogram is a quadrilateral whose opposite sides are parallel. It is a theorem that the quadrilateral *ABCD* is a parallelogram if and only if $\overrightarrow{AB}$ and $\overrightarrow{DC}$ have equal magnitudes and directions, that is, if and only if $\overrightarrow{AB} \cong \overrightarrow{DC}$. See Figure 11.

Exercises for Chapter I

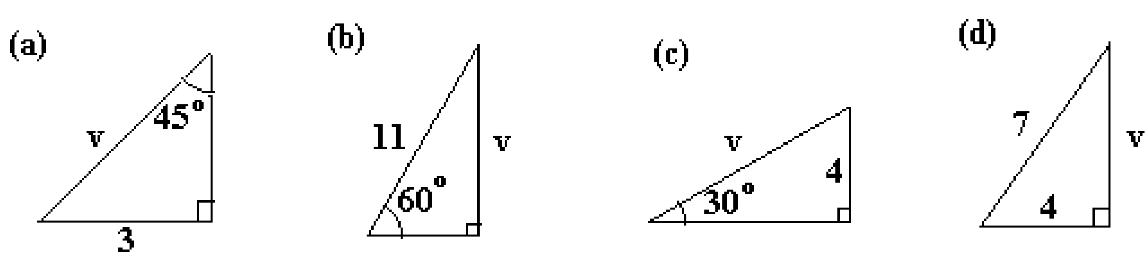
1. Classify each of the following angles as acute, right, obtuse, or not possible in a triangle:

$$-30^o,\ 0^o,\ \sqrt{37}\,^o,\ 90^o,\ 156^o,\ 180^o,\ 360^o.$$

2. Tell which of the following pairs of angles are possible in the same triangle and find the third angle in each such case:

   (a) $90^o, 90^o$; (b) $100^o, 85^o$; (c) $50^o, 60^o$; (d) $30^o, 140^o$.

3. Find (i) the exact value and (ii) a two decimal approximation (See Preface) for *v* in each of the following triangles:



4. Of the following triples, first identify those which represent sides of a triangle. Of those, select the right triangles and identify them. Then pick out the $45^o, 45^o, 90^o$ triangles and the $30^o, 60^o, 90^o$ triangles:

(a) $\sqrt{5},\ \sqrt{5},\ 10$

(b) 2, 3, 11

(c) 5, 12, 13

(d) 8, 9, 11

(e) $3,\ 3,\ 3\sqrt{2}$

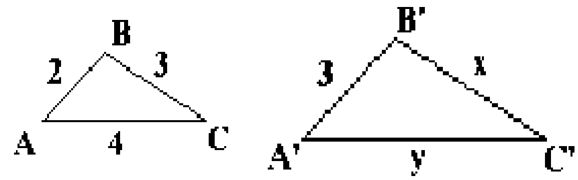
(f) $1,\ 2,\ \sqrt{5}$

(g) $1,\ \sqrt{2},\ \sqrt{3}$

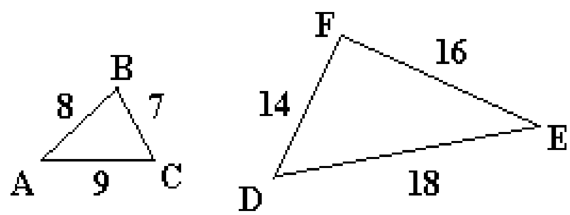
(h) $3\sqrt{3},\ 9,\ 6\sqrt{3}$

(i) 7.11, 9.48, 11.85

5. Given that ΔABC is similar to $\Delta A'B'C'$, find *x* and *y*.



6. Which angle in ΔDEF is equal to angle *A*?

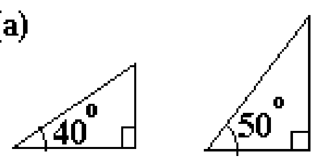


7 (a) Is a triangle with sides $\sqrt{3},\ 3,\ 2\sqrt{3}$ similar to one with sides $5,\ 10,\ 5\sqrt{3}$?

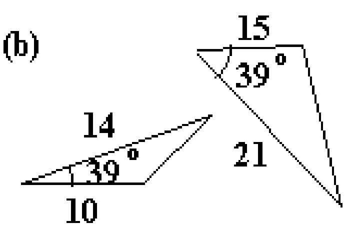
(b) Is a triangle with sides 23.472, 41.144, 51.256 similar to one with side 55.1592, 96.6884, 110.2004?

8. Which of the following pairs of triangles are similar? Justify your answers.

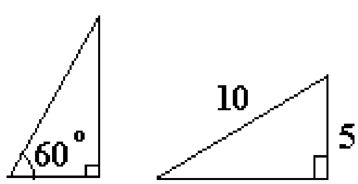
(a)



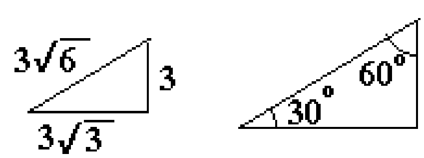
(b)



(c)



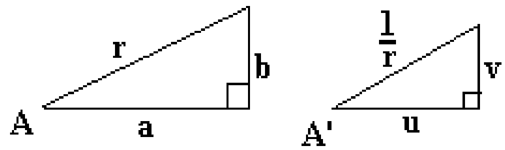
(d)



9. Given: angles A and A' are equal.

(a) Find u and v in terms of r, a, and b.

(b) If $a = 2.6$ and $b = 1.1$ find r, u, and v to 3 decimal places.



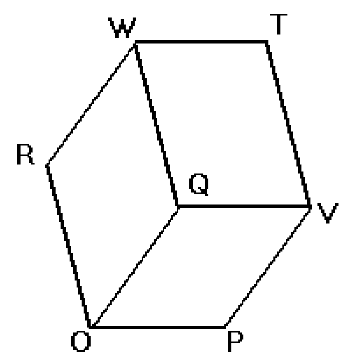
10. Are any two 30°, 60°, 90° triangles similar? Justify your answer.

11. Explain why a triangle whose sides are of length $\sqrt{a}$, $\sqrt{b}$, $\sqrt{a+b}$ is a right triangle. (Of course, $a > 0$ and $b > 0$.)

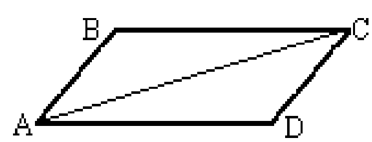
12. Complete the following table for converting certain degree measures to radians or, when read properly, radians to degrees:

θ in degrees	0°	30°	45°			120°		150°	180°
θ in radians	0	$\pi/6$		$\pi/3$	$\pi/2$		$3\pi/4$		

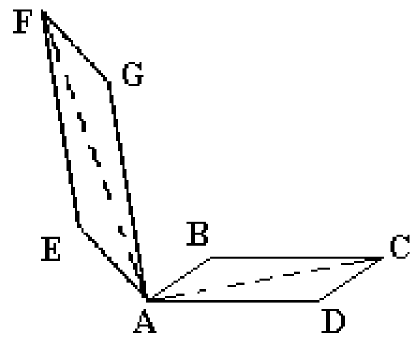
13. Given that $\vec{OQ} \cong \vec{PV} \cong \vec{RW}$ and that $\vec{VT} \cong \vec{OR}$, show that $\vec{WT} \cong \vec{OP}$.



14. Show that the quadrilateral $ABCD$ is a parallelogram if and only if ΔABC is similar to ΔCDA with $\triangle ACB = \triangle CAD$.



15. Prove that $AEFG$ is a parallelogram, given that $ABCD$ is a parallelogram, $\frac{AE}{AB} = \frac{AF}{AC} = \frac{AG}{AD}$, $\triangle DAC = \triangle GAF$, and $\triangle CAB = \triangle FAE$.



16. Use the methods depicted in Figure 9b and in Figure 9c in Section 4 above to show how to construct the angles given below with 30°, 60°, 90° and 45°, 45°, 90° triangles.

    (a) 105°; (b) 15°; (c) 225°; (d) 135°; (e) 195°.

17. Let $\triangle C = \pi/2$ in ΔABC. Let a, b, and c stand for the lengths of sides opposite $\triangle A$, $\triangle B$, and $\triangle C$, respectively. Find:

    (a) c when $a = 5$ and $b = 12$

    (b) b when $a = 8$ and $c = 17$

    (c) $\triangle A$ when $b = 4\sqrt{3}$ and $c = 8$

    (d) $\triangle B$ when $b = 5\sqrt{2}$ and $c = 10$.

18. Given a unit length, outline the construction with straightedge and compass of lengths of

    (a) 2/3

    (b) $\sqrt{5}$.