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Calculator Lesson 27

Vector Calculus

The calculator has very good tools for dealing with the derivatives of vector functions, but not so good for dealing with their integrals. Two of the three commands discussed in Lesson 8 for differentiating a scalar function work the same way for differentiating a vector function. They are the commands DERIV and DERVX. For the examples that follow the calculator should be in exact mode with flag 3 unchecked.

Suppose the position of a moving particle is given by the vector function $\mathbf{r} = \langle \cos(t) \quad \sin(t) \quad t \rangle$. (This path is called a helix.) Find the velocity and acceleration of the particle. The example that follows assumes the default variable is X. Enter the vector **r** by typing:

LS [] ‘ COS AS LS T RA RA SPC ‘ SIN AS LS

T RA RA SPC ‘ AS LS T ENTER.

Now find the velocity by using the DERIV command by pressing:

‘ AS LS T ENTER LS CALC F1-DERIV F2-DERIV

and we see the velocity is given by $\mathbf{v} = \langle -\sin(t) \quad \cos(t) \quad 1 \rangle$. Now go into MODE CAS and change the default variable to t. Next press ENTER three times to create three more copies of **v** on the stack. To find the acceleration we now press F3-DERVX and see that the acceleration is $\mathbf{a} = \langle -\cos(t) \quad -\sin(t) \quad 0 \rangle$. Now suppose we want to find the curvature of this path at the point $t = 1.5$. (NOTE: it is known that the curvature of this helix is .5 at any point, but we will pretend we don’t know that.) The formula for the curvature is given by $\kappa = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$. Our stack has **v** on levels 3 and 2, and **a** on level 1, so we proceed as follows:

LS MTH F1-VECTR F3-CROSS F1-ABS RA F1-ABS 3 Y^X ÷ ‘ AS LS

T RS = 1.5 ENTER RS ALG NXT F2-SUBST RS →NUM

and we see we get the answer .5 with a small round off error at the end.

Finally suppose we want to find the length of the helix from $t = 0$ to $t = 2\pi$. The formula for the length of a curve defined by the vector $\mathbf{r}(t)$ from $t = a$ to $t = b$ is given by $L = \int_a^b \|\mathbf{r}'(t)\| dt$. We still have a copy of **v = r’** on level 2 of the stack We saw in Lesson

14 that it is advisable to set the calculator to approximate mode when seeking definite integrals, so press RS(hold) →NUM to change the mode. Now press

$$\leftarrow \text{ LS ABS 0 ENTER RA LS } \pi \text{ 2 } \times \text{ RA ' AS LS T ENTER RS } \int$$

and see the answer 8.8857658763. This is equal to $2\sqrt{2}\pi$ as it should be.

The integral for the above example could be done using one of the methods from Lesson 14 because the length of a vector function is a scalar function. When we want to find the integral of a vector function, however, we must work a little harder.

Suppose a particle starts at the point (0, 1, -2) at time $t = 0$ and moves with velocity $\mathbf{v} = \left\langle -\sin(t) \quad \cos(t) \quad \dfrac{e^{t/\pi}}{\pi} \right\rangle$ and we wish to find the vector function for its location. The solution is $\mathbf{r} = \int \mathbf{v}(t)dt + \mathbf{c}$ where **c** is a constant vector that can be found using the given initial condition. We learned in Lesson 17 that the calculator mode must be exact for indefinite integration, so reset the mode. Now enter the vector and execute the INTVX command. We now see the answer

$$\begin{bmatrix} \cos(t) & \sin(t) & \dfrac{\dfrac{e^{\frac{t}{\pi}}}{\frac{1}{\pi}}}{\pi} \end{bmatrix},$$

which is correct but the third coordinate is not very attractive. If we press EVAL to try to simplify the third coordinate, nothing happens. To simplify this we must disassemble the vector, simplify the offending coordinate, then reassemble the vector. The following sequence gets the job done:

LS PRG F5-TYPE F1-OBJ→ RA EVAL RA F2-→ARRY.

We now see vector $\mathbf{r}_0(t) = \begin{bmatrix} \cos(t) & \sin(t) & e^{\frac{t}{\pi}} \end{bmatrix}$, which is certainly more attractive. To complete the problem we must use the given initial condition for find the constant vector **c.** This can be found by $\mathbf{c} = \langle 0, \quad 1, \quad -2 \rangle - \mathbf{r}_0(0)$. Save the vector function $\mathbf{r}_0$ with the name R because we will need it again. Now put the vector [0 1 -2] on the stack, bring back $\mathbf{r}_0$ and enter

RS ALG NXT ' AS LS T RS = 0 ENTER F2-SUBST RS →NUM -.

We now see that **c** = [-1 1 -3]. To complete the problem bring $\mathbf{r}_0$ back and press RA + we see our solution $\mathbf{r} = \begin{bmatrix} \cos(t)-1 & \sin(t)+1 & e^{\frac{t}{\pi}}-3 \end{bmatrix}$. Save this vector in the memory location where $\mathbf{r}_0$ was previously saved.

Now suppose we are interested in finding the displacement vector for this motion on the interval from $t = \pi$ to $t = 2\pi$. The solution is $\int_{\pi}^{2\pi} \mathbf{v}(t)dt$. Unfortunately, the calculator does not seem to have any direct way to evaluate the definite integral of a vector. In this case, since we know the indefinite integral we can use the Fundamental Theorem of Calculus to find $\int_{\pi}^{2\pi} \mathbf{v}(t)dt = \mathbf{r}(2\pi) - \mathbf{r}(\pi)$. Press VAR F1-R F1-R to get two compies of **r** on the stack. Now press:

RS ALG NXT ‘ AS LS T RS = 2 × LS π ENTER F2-SUBST RS

→NUM RA ‘ AS LS T RS = LS π ENTER F2-SUBST RS →NUM -.

Set the display to FIX 3 to make it easier to read and we see the result is [2.000 0.000 4.671].

The alternative to the procedure used above would be to disassemble the vector **v** and use the ROLL command to cycle through the elements and integrate them one at a time, then reassemble the vector. This would, if fact, be the only way to do it if one or more of the elements did not have an antiderivative.

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